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UNIVERSITY OF ALBERTA

MARTIN BOUNDARY FOR DISCRETE MARKOV CHAINS

BY

© *LAKHDAR AGGOUN*

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE.

DEPARTMENT OF STATISTICS AND APPLIED PROBABILITY

EDMONTON, ALBERTA

FALL, 1990

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ENTITLED *MARTIN BOUNDARY FOR DISCRETE MARKOV CHAINS*

SUBMITTED BY *LAKHDAR AGGOUN*

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE.

BYRON A. SCHMULAND

ATA AL-HUSSAINI

A. KAMAL

Date: September, 1990

Abstract

The boundary theory of Markov processes permits the investigation of the "final" behavior of the paths of such processes as the time tends to infinity. In this thesis we give explicit constructions of the Martin boundary for some discrete Markov chains. This construction is extended to the product of discrete Markov chains.

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CHAPTER 1

INTRODUCTION

The boundary theory of Markov processes permits the interpretation of the "final" behavior of their paths, that is, the behavior of X_n as the time n tends to infinity. An important application of boundary theory is the description of all harmonic and superharmonic positive functions connected with the process. This problem motivated the creation of the theory of the Martin boundary by Martin [16] who investigated the set of positive solutions of Laplace's equation in an arbitrary domain of Euclidean space. The probabilistic interpretation of Martin's results was proposed by Doob [9] for discrete Markov chains. These results are directly related to the Wiener process as was observed by Kakutani [28] who showed that if E is an equilibrium set, the value of the equilibrium potential at x is equal to the probability that a Brownian motion begun at x ever hits the set E. After that, it gradually became clear that in a certain sense Brownian motion and potential theory were really the same subject. In this thesis, we follow the approach of Kemeny, Snell and Knapp [1], who use Hunt's probabilistic approach to obtain a boundary which is more like Doob's than Martin's.

The second chapter contains definitions and notation on denumerable matrices and basic probability theory.

The third chapter covers properties of discrete Markov chains which will be used in the sequel.

Chapter four introduces the potential theory for transient Markov chains. Definitions and theorems on the concepts of regular functions and measures, potentials, charges, capacities, etc. for Markov chains are given in sections 4.1 and 4.2. Section 4.3 deals with extended chains and h-processes. The definition of the

Martin boundary and the Poisson-Martin representation are found in sections 4.4 and 4.5 respectively.

Chapter five starts with a brief history of the evolution of the theory of Martin boundaries which is followed by explicit constructions of the boundary for random walks, renewal chains, birth and death chains, Galton-Watson processes etc.

Chapter six extends the construction to products of Markov chains where in some situations it turns out that it is easier to obtain the Feller boundary when both boundaries coincide.

CHAPTER 2

PRELIMINARIES

2.1. Denumerable matrices

Let M and N be two non-empty denumerable sets. A denumerable (or countably infinite) matrix is a function from $M \times N$ to the extended real line $\mathbb{R} \cup \{+\infty, -\infty\}$, in other words the value of the function on the pair (m,n) or the m-nth entry of the matrix is a number in the extended real line.

A matrix is said to be *non-negative (positive)* if all its entries are non-negative (positive).

A matrix whose first index set (second index set) contains only one element is called a *row vector* (a *column vector*).

If A is a matrix defined on $M \times N$, then the matrix A^T called the *transpose* of A and is defined on $N \times M$ by $(A^T)_{mn} = A^T{}_{nm}$.

The column vector all of whose entries are equal to 1 is denoted $\mathbb{1}$. A matrix all of whose entries are ones is denoted by E.

A *bounded* vector is a vector all of whose entries are bounded by some real number.

Let A and B be two matrices with the same domain of definition , then A=B if and only if $A_{mn} = B_{mn}$ $\forall (m,n) \in M \times N$, $A < B$ if and only if $A_{mn} < B_{mn}$, $A + B$ is well-defined if and only if A and B are both defined on the same $M \times N$ and each entry $(A + B)_{mn} = A_{mn} + B_{mn}$ is well defined.

Let A be defined on $K \times M$, and let B be defined on $M \times N$. Suppose the sums $\sum_m A_{km} B_{mn}$ are well-defined for all k and n. The matrix $C = AB$ is said to be

well-defined on $K \times N$ and its entries are given by

$$C_{kn} = \sum_m A_{km} B_{mn}.$$

If A, B and C are matrices such that AB, AC and AB + AC are well defined then $A(B + C) = AB + AC$.

Non-negative matrices associate under multiplication.

Matrix inequalities of the form $A \le B$ $(A < B)$ are preserved when both sides of the inequality are multiplied by a non-negative matrix (positive matrix).

A matrix P with non-negative entries is called a *stochastic matrix* if $P\mathbb{1} = \mathbb{1}$, and is called substochastic if $P\mathbb{1} \le \mathbb{1}$.

Let π be a non-negative row vector and $\{f^{(k)}\}$ a sequence of non-negative column vectors, then

$$\pi(\liminf f^{(k)}) \le \liminf (\pi f^{(k)}),$$

so if $f = \lim f^{(k)}$ exists, then $\pi f \le \liminf (\pi f^{(k)})$.

This is the discrete version of Fatou's lemma. The discrete version of the monotone convergence theorem is given below.

If $\{f^{(k)}\}$ is non-decreasing and converges entry - by - entry to f we have

$$\pi f = \lim \pi f^{(k)}.$$

It follows from this result that $\pi \sum f^{(k)} = \sum \pi f^{(k)}$.

Let $\pi \ge 0$ be a row vector such that $\pi\mathbb{1}$ is finite. If $\{f^{(k)}\}$ is a sequence of column vectors such that $|f^{(k)}| \le c\mathbb{1}$ and $f = \lim f^{(k)}$ exists, then

$$\pi f = \lim \pi f^{(k)}.$$

If $\{\pi^{(k)}\}$ is a sequence of non-negative row vectors converging to π, if f is a column vector such that $|f| \le c\mathbb{1}$ for some c, and if

$\pi\mathbb{1} = \lim \pi^{(k)}\mathbb{1}$, with $\pi\mathbb{1}$ finite, then $\pi f = \lim \pi^{(k)} f$.

Let $\{\pi^{(k)}\}$ be a sequence of row vectors converging to π and satisfying $|\pi^{(k)}|_1 \leq M$. Suppose f is a column vector such that for any $\delta > 0$ only finitely many entries of f have absolute value greater than δ. Then $\pi f = \lim \pi^{(k)} f$.

2.2. Basic probability concepts

Let Ω be a set of points. A non-empty class $\mathcal{B}$ of subsets is called a σ-*algebra* if $\mathcal{B}$ is closed under complementation and countable unions.

If $\{B_\alpha, \alpha \in G\}$ is a non-empty class of subsets of Ω, then $\sigma(B_\alpha, \alpha \in G)$ denotes the smallest σ-algebra containing each B_α.

A function $P : \mathcal{B} \to [0, 1]$ is called a *probability measure* if

(i) $P(\Omega) = 1$

(ii) If $\{B_n\}$ is a countable sequence of disjoint sets in $\mathcal{B}$, then $P(\cup B_n) = \sum P(B_n)$.

The triple $(\Omega, \mathcal{B}, P)$ is called a probability space. A set $B \in \mathcal{B}$ is sometimes referred to as an *event*.

If $X : \Omega \to \mathbb{R}$ is a function we let $\sigma(X) = \sigma(\{\omega / X(\omega) \leq x\}, x \in \mathbb{R})$. If X is $\mathcal{B}$-measurable, that is, if $\sigma(X) \subseteq \mathcal{B}$, then we call X a *random variable* (abbrev. r.v.).

For $C \in \mathcal{B}$ we define 1_C, the indicator function of C, as follows

$1_C(\omega) = 1$ if $\omega \in \mathcal{B}$

$= 0$ otherwise.

For a sequence $B_1, B_2, \ldots$ of events, define the event

$$\limsup B_n = \bigcap_{n \geq 1} \bigcup_{k \geq n} B_k \qquad (2.1)$$

This event is the limit superior of the sequence $\{B_n\}$. Now ω lies in (2.1) if and only if for each n there exists some $k \geq n$ for which $\omega \in B_k$; in other words ω lies in (2.1) if and only if it lies in infinitely many of the B_n. When n expresses time (2.1) is frequently written

$$\limsup B_n = [B_n \text{ i.o.}] \qquad (2.2)$$

where i.o. stands for "infinitely often".

LEMMA 2. 2. 1. (Borel-cantelli) If $\sum P(B_n)$ converges then $P[\limsup B_n] = 0$.

LEMMA 2. 2. 2. (Borel-cantelli) If B_n is a sequence of independent events and $\sum P(B_n)$ diverges, then $P[\limsup B_n] = 1$.

A *stopping time* is an r.v. $T : \Omega \to \mathbb{N} \cup \{\infty\}$ such that $\{\omega / T(\omega) = n\} \in \mathcal{B}_n \ \forall\, n \in \mathbb{N}$ where $\{\mathcal{B}_n\}$ is an increasing sequence of sub-σ-algebras of $\mathcal{B}$.

If $\{X_n\}$ is a sequence of random variables and T a stopping time, the random variable X_T, is defined as follows :

$$X_T = X_{T(\omega)}(\omega) \qquad \text{if } T(\omega) < \infty$$

$$= 0 \qquad \text{otherwise.}$$

The σ-algebra associated with T, denoted $\mathcal{B}_T$, is defined as

$$\mathcal{B}_T = \{C \in \sigma(\cup \mathcal{B}_n) : C \cap \{T=n\} \in \mathcal{B}_n \ \forall\, n \in \mathbb{N}\}.$$

The set of all stopping times is given the natural ordering (<), if U and V are stopping times and $P(U < V) = 0$, then $V < U$.

A random variable is called simple if there are finitely many real numbers $x_1, \ldots, x_n$ such that $\sum P(X=x_i) = 1$.

The integral of a simple random variable is defined as

$$\int X \, dP = \sum x_i P(X=x_i).$$

If X is a non-negative random variable , its integral is defined as

$$\int X \, dP = \lim \int X_n dP,$$

where $\{X_n\}$ is a sequence of simple random variables which increases pointwise to X. If the above limit is finite, X is said to be *integrable*. If both its positive and negative parts, $X^+ = X\,1_{\{X \geq 0\}}$ and $X^- = -X\,1_{\{X \leq 0\}}$ are integrable, we define

$$\int X\ dP = \int X^+\ dP - \int X^-\ dP.$$

The expected value of X is written E(X), and is equal to $\int X\,dP$.

For $C \in \mathcal{B}$, we write

$$\int_C X\ dP = \int X\,1_C\ dP.$$

A sequence of random variables $\{X_n\}$ is said to converge *almost surely* (a.s) if

$$P[\ \lim X_n(\omega) \text{ exists and is finite }] = 1.$$

$\{X_n\}$ converges to X in L^1 if $E[\ |X_n - X|\ ] \to 0$ as $n \to \infty$.

$\{X_n\}$ converges to X *in probability* if for each $\varepsilon > 0$ the sequence

$$P[\ |X_n - X| > \varepsilon\ ] \to 0 \text{ as } n \to \infty.$$

$\{X_n\}$ is said to converge to X *weakly in* L^1 if

$$E(X_n Y) \to E(XY) \text{ for each } Y \in L^\infty.$$

2.3. Conditional probability and expectation

If A, B are two events, then we define the probability of A given B as

$$P(A / B) = \frac{P(A \cap B)}{P(B)},$$

provided $P(B) > 0$. Otherwise $P(A / B)$ is left undefined. More generally we have the concept of conditional expectation which we now define.

Let $X \in L^1$ and $\mathcal{A}$ be a sub σ-algebra of $\mathcal{B}$. By the Radon-Nikodym theorem there exists a $\mathcal{A}$-measurable random variable denoted by $E(X / \mathcal{A})$ uniquely determined except on an event of probability zero, such that

$$\int_A X \, dP = \int_A E(X / \mathcal{A}) \, dP \quad \text{for all } A \in \mathcal{A}.$$

$E(X / \mathcal{A})$ is called the *conditional expectation* of X given $\mathcal{A}$. If A is an event, then we write $P(A / \mathcal{A})$ as short hand for $E(1_A / \mathcal{A})$.

The following is a list of classical results.

If $\mathcal{A}_1$ and $\mathcal{A}_2$ are two sub σ-algebras of $\mathcal{B}$ such that $\mathcal{A}_1 \subseteq \mathcal{A}_2$, then

$$E(E(X / \mathcal{A}_1) / \mathcal{A}_2) = E(E(X / \mathcal{A}_2) / \mathcal{A}_1) = E(X / \mathcal{A}_1).$$

If $X, Y, XY \in L^1$ and Y is $\mathcal{A}$- measurable then

$$E(XY / \mathcal{A}) = YE(X / \mathcal{A}).$$

If X and Y are independent, then

$$E(X / \sigma(Y)) = E(X).$$

If $\{X_n\}$ is a non- decreasing sequence of non-negative random variables such that X_n converges to X a.s. and $X \in L^1$, then

$E(X_n / \mathcal{A})$ converges to $E(X / \mathcal{A})$ a.s.

Let g be a convex function on $\mathbf{R}$ and let $X \in L^1$ such that $g(X) \in L^1$, then

$$g(E(X / \mathcal{A})) \leq E(g(X) / \mathcal{A}).$$

CHAPTER 3

MARKOV CHAINS

3.1. Basic definitions and properties

Let $(\Omega, \mathcal{B}, P)$ be a probability measure space, and S be a countable set which we call the *state space*. Then a *Markov chain with values* in S is a sequence of random variables $X_0, X_1, \ldots$ such that :

(i) X_i; $i = 0, 1, 2, \ldots$ are $\mathcal{B}$ - measurable,

(ii) each X_i takes on its values in S,

(iii) if $(x_0, x_1, \ldots, x_{n+1}) \in S^{n+1}$, then

$$P(X_{n+1} = x_{n+1} / X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_0 = x_0) = P(X_{n+1} = x_{n+1} / X_n = x_n)$$

if both sides are defined. This last equation is called the *Markov property*, and it means (roughly) that the future behaviour of the process $\{X_n\}$ only depends on the past through the present.

A Markov chain $\{X_n\}$ has *stationary transitions* if

$$P(X_{n+1} = j / X_n = i) = P(X_1 = j / X_0 = i) \quad \forall\ n.$$

This means that the probability of a transition from state i to state j remains constant in time. In the remainder of the thesis we will only consider chains with stationary transitions.

The quantity $P_{ij} := P(X_1 = j / X_0 = i)$ is called the *transition function* and since S is countable we can think of $P = \{P_{ij}, (i, j) \in S \times S\}$ as matrix which has the property that the row sums are equal to one. Such a matrix is called a *stochastic matrix*.

If π denotes the distribution of X_0 (also called the *initial distribution*), then it is easily seen that the distribution of X_m is given by

$$\pi_m = \pi P^m.$$

If we consider times $0 \leq m < u < n$, then the Markov property shows that

$$P(X_n = k / X_m = j) = \sum_i P(X_n = k, X_u = i / X_m = j)$$

$$= \sum_i P(X_n = k / X_u = i, X_m = j) P(X_u = i / X_m = j) \qquad (3.1)$$

$$= \sum_i P(X_n = k / X_u = i) P(X_u = i / X_m = j).$$

Define $P_{jk}(n-m) = P(X_n = k / X_m = j)$ i.e., the probability of making a transition from state j to state k in (n-m) units of time. Set $P(n-m) = \{P_{jk}(n-m)\}$ for $(j, k) \in S \times S$. This is a stochastic matrix and (3.1) gives us the Chapman-Kolmogorov equations,

$$P(n-m) = P(n-u)\, P(u-m).$$

By induction we have $P(n) = P^n$, where $P = P(1)$.

DEFINITION 3. 1.

a) P_i and E_i refer to $P(. / X_0 = i)$ and $E(. / X_0 = i)$ respectively.

b) T^i is the hitting time of state i, that is,

$T^i(\omega) = \min \{n > 0; X_n(\omega) = i\}$ if such an n exists

$= +\infty$ otherwise.

c) $H_{ij} = P_i(X_n \text{ visits } j, n \geq 0)$

$\underline{H}_{ij} = P_i(X_n \text{ visits } j, n \geq 1)$

$N_{ij} = E_i(\text{number of visits to state } j)$

$\underline{N}_{ij} = E_i(\text{number of visits to state } j, n \geq 1)$

LEMMA 3. 2. For $r \geq 0$, $m \geq 1$ and $(x_1, ..., x_m) \in S^m$ we have

$$P(X_{r+m} = x_m, \ldots, X_{r+1} = x_1 / X_r) = P_{X_r}(X_m = x_m, \ldots, X_1 = x_1).$$

PROOF: Using the stationarity and the Markov property we can write, for each $y \in S$,

$$P(X_{r+m} = x_m, \ldots, X_{r+1} = x_1 / X_r = y) = P(X_{r+m} = x_m / X_{r+(m-1)} = x_{m-1})$$

$$P(X_{r+(m-1)} = x_{m-1} / X_{r+(m-2)} = x_{m-2}) \ldots P(X_{r+1} = x_1 / X_r = y)$$

$$= P(X_m = x_m / X_{m-1} = x_{m-1}) \ldots P(X_1 = x_1 / X_0 = y)$$

$$= P(X_m = x_m , X_{m-1} = x_{m-1} , \ldots , X_1 = x_1 / X_0 = y)$$

$$= P_y(X_m = x_m , \ldots , X_1 = x_1).$$

Since

$$P(X_{r+m} = x_m , \ldots , X_{r+1} = x_1 / X_r)$$

$$= \sum_y P(X_{r+m} = x_m , \ldots , X_{r+1} = x_1 / X_r = y) 1_{\{X_r = y\}}$$

we have the result.

LEMMA 3. 3. If T is a stopping time and $\{X_n\}$ is a Markov chain, for any $m \in \mathbb{N}$ and any subset B of S^{m+1} we have

$$P((X_T, X_{T+1}, \ldots , X_{T+m}) \in B / \mathcal{B}_T) = P_{X_T}((X_0, \ldots , X_m) \in B) \quad \text{a. s.}$$

on the set $\{T < \infty\}$. This property is known as the *strong Markov property*.

PROOF: Since S^{m+1} is countable it suffices to consider the case $B = (x_0, \ldots , x_m)$. Then

$$P(X_T = x_0, \ldots , X_{T+m} = x_m / \mathcal{B}_T)\, 1_{\{T<\infty\}} = \sum_{r=0}^{\infty} P(X_T = x_0, \ldots , X_{T+m} = x_m , T = r / \mathcal{B}_T).$$

Let $A \in \mathcal{B}_T$, by definition of $\mathcal{B}_T$ we have

$$A \cap \{T = r\} \in \sigma(X_0, \ldots , X_r) \quad \text{so}$$

$$P(X_T = x_0, \ldots , X_{T+m} = x_m , T = r, A) = P(X_r = x_0, \ldots , X_{r+m} = x_m , T = r, A) =$$

$$\int_{A \cap \{T=r\}} P(X_r = x_0, \ldots , X_{r+m} = x_m / \sigma(X_0, \ldots , X_r))dP =$$

$$\int_{A \cap \{T = r\}} P(X_r = x_0, \ldots , X_{r+m} = x_m / X_r)dP = \int_{A \cap \{T=r\}} P_{X_r}(X_0 = x_0, \ldots , X_m = x_m)dP,$$

where lemma 3.2 was used in the last step. Adding over r gives the result.

The strong Markov property can be stated using the *shift operator* θ which is defined from $\Omega = S^{\mathbb{N}}$ into itself by

$$\theta(\{\omega_0, \omega_1, ..., \omega_n, ... \}) = \{\omega_1, \omega_2, ... \}.$$

We also set $\theta_1 = \theta$ and define, for $p \geq 1$, $\theta_{p+1}(\omega) = \theta(\theta_p(\omega))$. Notice that we have

$$X_n(\theta_p(\omega)) = X_{n+p}(\omega), \quad \omega \in \Omega.$$

The Markov property is expressed as follows:

for any positive r.v. Y, we have almost surely

$$E_\pi (Y o \theta_n / \mathcal{B}_n) = E_{X_n}(Y), \tag{3.2}$$

where $\mathcal{B}_n = \sigma\{X_m, m \leq n\}$ and $E_{X_n}(Y) = E_i(Y)$ on $\{X_n = i\}$.

The strong Markov property is expressed by replacing n by any $\{\mathcal{B}_n\}$-stopping time τ, i.e.,

$$E_\pi (1_{[\tau<\infty]} Y o \theta_\tau / \mathcal{B}_\tau) = 1_{[\tau<\infty]} E_{X_\tau}(Y) \quad P_\pi\text{-a.s.}, \tag{3.3}$$

where $E_{X_\tau}(Y) = E_i(Y)$ on $\{\omega: X_{\tau(\omega)}(\omega) = i\}$

A function $H: \Omega \to \mathbb{R}$ is called invariant if it is constant under shift, that is, $H(\omega) = H(\theta(\omega)) \quad \forall \omega \in \Omega$.

3.2. Classification of states

A set A in S is *closed* with respect to a Markov chain with transition function P_{ij}, , if

$$\sum_{j \in A} P_{ij} = 1 \quad \forall i \in A.$$

This means that once the chain enters A, it cannot exit from A.

A Markov chain is *indecomposable* or irreducible if there are no two disjoint closed sets.

The following results are proven in [1]

a) $\underline{N} = PN$,

b) $\underline{H} = PH$,

c) $N_{ij} = H_{ij} N_{jj}$,

d) $\underline{N}_{ij} = \underline{H}_{ij} N_{jj}$,

e) $N_{ii} = 1 + \underline{H}_{ii} N_{ii}$

f) $N = \sum_{k=0}^{\infty} P^k$.

Here N, $\underline{N}$, H, $\underline{H}$ are all matrices whose entries are defined in Definition 1.

THEOREM 3. 4.

(a) If $\underline{H}_{jj} = 1$, then $P_j(X_n = j \text{ i. o.}) = 1$ and $N_{jj} = \infty$. In this case state j is said to be *recurrent.*

(a) If $\underline{H}_{ij} < 1$, then $P_j(X_n = j \text{ i. o.}) = 0$ and $N_{jj} = (1-\underline{H}_{jj})^{-1} < \infty$. In this case state j is said to be *transient.*

PROOF: (a) $\{X_n = j \text{ i. o.}\}^c = \cup A_n$ for $n \geq -1$ where

$A_{-1} = \{ X_k \neq j \ \forall k \geq 0\}$,

$A_n = \{X_n = j, X_{n+k} \neq j \ \forall k \geq 1\}$, $n = 0, 1, 2, \ldots$

Since all the A_n are disjoint, and $P_j(A_{-1}) = 0$, we have

$$P_j(X_n = j \text{ i. o.}) = 1 - \sum_{n=0}^{\infty} P_j(A_n). \tag{3.4}$$

Now, by stationarity, for $n \geq 0$,

$P_j(A_n) = P(X_{n+1} \neq j, \ldots / X_n = j) P_j(X_n = j) = P_j(X_1 \neq j, \ldots) P_j(X_n = j)$.

Therefore

$$P_j(A_n) = P_j(A_0)\, P_j(X_n = j). \tag{3.5}$$

If $\underline{H}_{jj} = 1$, then $P_j(A_0) = 0$ and therefore $P_j(A_n) = 0$. From (3.4) we see that $P_j(X_n = j \text{ i. o.}) = 1$, and using the Borel-Cantelli lemma we find that $\sum_{n=1}^{\infty} P_j(X_n = j) = \infty$. Thus $N_{jj} = \infty$.

(b) Since $\underline{H}_{jj} = 1 - P_j(A_0)$, if $\underline{H}_{jj} < 1$, we have $P_j(A_0) > 0$.

Then (3.4) and (3.5) imply

$$1 - P_j(X_n = j \text{ i. o.}) = \sum_{n=0}^{\infty} P_j(A_n) = \sum_{n=0}^{\infty} P_j(A_0)\, P_j(X_n = j)$$

$$= P_j(A_0) \sum_{n=0}^{\infty} P_j(X_n = j).$$

Since $P_j(A_0) > 0$, then $\sum_{n=0}^{\infty} P_j(X_n = j) < \infty$ hence

$P_j(X_n = j \text{ i. o.}) = 0$ by Borel-Cantelli lemma.

Hence

$$\underline{H}_{jj} = 1 - P_j(A_0) = 1 - \frac{1}{\sum_{n=0}^{\infty} P_j(X_n = j)} = 1 - \frac{1}{N_{jj}}$$ which gives the result.

Let R_k^j denote the time of the k-th return to j and $T^j = T_1^j = R_1^j$, $T_k^j = R_k^j - R_{k-1}^j$ for $k > 1$. The random times $T_1^j, T_2^j \ldots$ are called the *recurrence times* for j. Using the strong Markov property it is seen that if j is recurrent, then $T_1^j, T_2^j, \ldots$ are i.i.d random variables under P_j.

A recurrent state j is said to be *positive recurrent* if $E_j(T_1^j) < \infty$, which means that the mean time to return to state j is finite.

It is said to be *null recurrent* if $E_j(T_1^j) = \infty$, which means that the mean time to return to state j is infinite.

Define $d_j = \text{g. c. d. } (n : (P^n)_{jj} > 0,\ n \geq 1)$. Then the state j is called *aperiodic* if $d_j = 1$ and *periodic*, with *period* d_j, otherwise.

Let $i, j \in S$

If $H_{ij} > 0$ write $i \to j$.

If $H_{ij} H_{ji} > 0$ we write $i \leftrightarrow j$ and in this case we say i and j *communicate*, in other words, it is possible for the chain X_n to make a trip from state i to state j, and then return to i.

THEOREM 3. 5. If $i \leftrightarrow j$, then j is transient (respectively null recurrent, positive recurrent) if and only if i is.

THEOREM 3. 6. For an aperiodic chain

$$P_j(X_n = j) \rightarrow \frac{1}{E_j(T^j)} \qquad \text{if } E_j(T^j) < \infty$$

$$\rightarrow 0 \qquad \text{if } E_j(T^j) = \infty$$

THEOREM 3. 7. Let C be an aperiodic recurrent class of states. Then for $j, k \in C$,

$$(P^n)_{kj} \rightarrow 0 \qquad \text{if } j \text{ is null recurrent}$$

$$(P^n)_{kj} \rightarrow \mu_j = \frac{1}{E_j(T^j)} \qquad \text{if } j \text{ is positive recurrent.}$$

PROOF: The result is true for $j = k$ by Theorem 3. 6. Now the Chapman-Kolmogorov equations give us

$$(P^n)_{kj} = \sum_{m=1}^{n} P_k(T^j = m)\,(P^{n-m})_{jj}.$$

By bounded convergence we have

$$(P^n)_{kj} \rightarrow \sum_{m=1}^{\infty} P_k(T^j = m)\mu_j = \mu_j P_k(T^j < \infty) = \mu_j \qquad \text{if } E_j(T^j) < \infty,$$

$$\rightarrow 0 \qquad \text{otherwise.}$$

THEOREM 3. 8. If all the states communicate, and are positive recurrent, then μ is the unique stationary probability distribution. If they are null, there is no stationary probability distribution.

PROOF: If π is a stationary probability distribution, then $\pi = \pi P$ so by induction

$\pi = \pi P^n \quad \forall n$ that is,

$\pi(j) = P_\pi(X_n = j) = \sum_k \pi(k) P_k(X_n = j).$

By bounded convergence we can write

$\pi(j) = \lim \sum_k \pi(k) P_k(X_n = j) = \sum_k \pi(k)\mu_j = \mu_j \quad \forall j.$

If the chain is null recurrent, then

$\pi(j) = \sum_k \pi(k) P_k(X_n = j) \rightarrow 0,$

So $\pi(j) = 0 \quad \forall j$ and there is no stationary probability distribution.

For the positive recurrent case, fix any k, then by Fatou's lemma,

$$\sum \mu_j = \sum_j \lim (P^n)_{kj} \leq \liminf \sum_j (P^n)_{kj} = 1.$$

Also $\mu_j = \lim_n P_k(X_{n+1} = j) = \lim_n \sum_i P_k(X_n = i) P_{ij} \geq \sum_i \mu_i P_{ij}$,

so $\mu_j \geq \sum_i \mu_i P_{ij} \quad \forall j$.

Hence

$$\sum \mu_j \geq \sum_j \sum_i \mu_j P_{ij} = \sum_i \mu_i \sum_j P_{ij} = \sum_i \mu_i.$$

Therefore, since the non-negative numbers $\mu_j - \sum_i \mu_i P_{ij}$ add up to zero, they are all zero. Hence $\mu_j = \sum_i \mu_i P_{ij}$, so μ is a stationary distribution.

Thus $\{ \frac{\mu_j}{\sum \mu_i} \}$ is a stationary probability distribution, and by uniqueness

$\frac{\mu_j}{\sum \mu_i} = \mu_j \quad \forall j$ in other words $\sum \mu_i = 1$.

CHAPTER 4

POTENTIAL THEORY FOR TRANSIENT MARKOV CHAINS

4.1. Introduction

The marriage of two major branches of theoritical physics, namely the probabilistic theory of random processes and potential theory was foreshadowed to 1928 by the work of Richard Courant, K. O. Friedrichs and H. Lewy in Germany. What this means is that all the main problems and features of classical potential theory have a Mathematical counterpart in the theory of Brownian motion; the main connection between the two theories being made by way of the central problem of potential theory which is called the Dirichlet problem and which is stated as follows: Let D be a domain in euclidean space with a smooth enough boundary denoted ∂D, and f a function continuous on ∂D. Find a function u defined on $\bar{D}$ such that $\sum \frac{\partial^2 u}{\partial x_i}$ $\equiv \Delta u = 0$ on D and $u = f$ on ∂D. The function u is then said to be harmonic. Potential theory is the mathematics of equilibrium. It studies harmonic functions, which arise whenever one has a homogeneous medium in a state of equilibrium. As an example, the distribution of heat in a homogeneous solide body at thermal equilibrium is a physical problem that involves a harmonic function. Since the temperature at any given point in such a body does not change with time, the temperature at that point must equal the average temperature over the surface of a small surrounding sphere. In other words, the temperature is a harmonic function. Now, knowing the temperature at all points on the surface of the body, can one compute the interior temperature? This is a Dirichlet problem. The first one to see that this problem can be solved by means of the probabilistic theory of Brownian motion was Kakutani. His solution to Dirichlet problem is given by:

$$u(x) = E_x(f(B_\tau)) \quad \forall x \in \bar{D}$$

where, $\tau = \inf\{t > 0: B_\tau \in \partial D\}$ and B_t is a Brownian motion.

Later, the mathematical equivalence of the two theories has been fully exploited by mathematicians including j. Doob, G. Hunt, M. Kac, E.B. Dynken, P.A. Meyer, S. Kakutani, K. Ito, etc. The discrete part of the theory, including the theory of Martin exit boundary for a Markov chain, was established by Doob and generalized by Hunt. Many works appeared thereafter.

This chapter contains some of the classical potential theory concepts as translated in the language of Markov chains. For more details see Chapters 7 and 8. of Kemeny, Snell and Knapp [1].

Let P be the transition matrix for a transient Markov chain. A column vector f is said to be a P-*regular function* or simply a *regular function* if $f = Pf$; it is *superregular* if $f \geq Pf$; it is *subregular* if $f \leq Pf$. The function f is said to be regular at a point j if $f_j = (Pf)_j$.

A non-negative row vector π is a *regular measure* if $\pi = \pi P$; it is *superregular* if $\pi \geq \pi P$ and *subregular* if $\pi \leq \pi P$.

Given a transition matrix P and a positive finite-valued P-superregular measure α, we define the α-*dual* of P to be

$$\hat{P}_{ij} = \frac{\alpha_j}{\alpha_i} P_{ji}.$$

Let D be the diagonal matrix with diagonal entries $1/\alpha_i$. We note that

$$\hat{P} = D\, P^T D^{-1}.$$

It can be shown that $\hat{P}$ is a transition matrix and that if P is transient so is $\hat{P}$.

We also define the α-duals of functions and measures as follows:

$$\hat{f} = f^T D^{-1}$$

$$\hat{\mu} = D\mu^T.$$

Note that duality interchanges functions and measures.

THEOREM 4. 1. If X_n is a Markov chain with transition matrix P and if f is a superregular function, then $f(X_n)$ is a supermartingale with respect to the σ-algebras $\sigma(X_0, \ldots, X_n)$.

PROOF: $E(f(X_{n+1}) / \sigma(X_0, \ldots, X_n)) = E_{X_n}(f(X_1)) = \sum f_j P_{ij} \leq f_i$ on the set $\{X_n = i\}$, therefore $E(f(X_{n+1}) / \sigma(X_0, \ldots, X_n)) \leq f(X_n)$ a.s. and we have the result.

Let us recall the matrix $N = \sum P^n$. If μ is a measure such that $\mu \mathbb{1} < \infty$, then $\nu = \mu N$ is well defined and said to be the *potential measure* of the *left charge* μ. If f is a function such that N f is well defined and finite, then g = N f is the *potential function* of the right charge f. A potential of a non-negative function or measure is a *pure potential*, which is always superregular. The *support* of a potential is defined to be the support of its charge.

THEOREM 4. 2. (Riesz decomposition) Let h be a non-negative superregular function. Then

$$h = Nf + r$$

where $f \geq 0$ and r is regular.

PROOF: Since h is superregular, the sequence $\{P^n h\}$ is decreasing, i.e., $h \geq Ph \geq P^2h \geq \ldots \geq 0$. Since these functions are bounded from below by 0, $r = \lim P^n h$ exists.

But $Pr = P \lim P^n h = \lim P^{n+1} h = r$ by the monotone convergence theorem. Hence r is regular. Also

$$h = (h - Ph) + (Ph - P^2h) + \ldots + (P^{n-1}h - P^n h) + P^n h.$$

Taking the limit in n gives

$$h = \sum_{n=0}^{\infty} P^n(h - Ph) + r = Nf + r \quad \text{and} \quad f = (I - P)h \geq 0.$$

THEOREM 4. 3. If H is an a.s. invariant function with $E_x(|H|) < \infty$ for each x then, $h(x) := E_x(H)$ is a regular function.

PROOF: Using the fact that H is shift invariant, and using the Markov property (3.2), we obtain

$$\begin{aligned} h(x) = E_x(H) &= E_x(E_x(H/\mathcal{B}_1)) \\ &= E_x(E_x(H\circ\theta_1/\mathcal{B}_1)) \\ &= E_x(E_{X_1}(H)) \\ &= E_x(h(X_1)) \\ &= \sum h(y)P_{xy}. \end{aligned}$$

THEOREM 4. 4. If P is the transition matrix of a Markov chain, then there exists a strictly positive superregular measure.

PROOF: Let $\alpha = \sum_{i\geq 0} c_i N_i$ $\quad (\sum c_i < \infty,\ c_i > 0)$,

where N_i is the i^{th} row of the matrix N.

Then α is finite-valued since $N_{ij} \leq N_{jj}$, and

$$(\alpha P)_k = \sum_i \sum_j c_i N_{ij} P_{jk} = \sum c_i (NP)_{ik} \leq \sum_i c_i N_{ik} = \alpha_k.$$

Hence α is superregular . But $N_{ii} > 0 \ \ \forall i$ so $\alpha > 0$.

For any Markov chain P we define a *hitting vector* h^E and an *escape* vector e^E by

$h_i^E = P_i\{$process eventually reaches the subset E$\}$

and

$e_i^E = P_i\{$process goes on first step from E to S-E and then never returns to E$\}$.

We have the following relations between h^E and e^E.

(1) $h^E = e^E + Ph^E$ and hence $e^E = (I - P)h^E$.

(2) If $E \subset F$, then $h^E \leq h^F$.

4.2. Equilibrium sets and capacities

DEFINITION 4. 5. A set E is an *equilibrium set* for P if there is a pure potential which assumes the value 1 at every point of E and which has support in E. Such a potential is called an equilibrium potential for E. A set E is a *dual equilibrium* set for P if there is a pure potential measure with support in E which equals α on E.

We proceed to give two characterizations of equilibrium potentials.

PROPOSITION 4. 6. A set E is an equilibrium set if and only if

(1) $\alpha e^E < \infty$ and

(2) for any starting distribution the set E is entered only finitely often a.s.

When E is an equilibrium set, the hitting vector h^E is the *unique* equilibrium potential and its charge is the escape vector e^E.

COROLLARY All finite sets are equilibrium sets.

DEFINITION 4. 7. If E is an equilibrium set, the *capacity* of E is defined by $C(E) = \alpha e^E = \eta^E \mathrm{II}$, where η^E is the dual of e^E.

PROPOSITION 4. 8. If $\hat{N}$, the α-dual of N, has columns which tend to zero, then a set E for which the supremum of the capacities of its finite subsets is finite is an equilibrium set, and the supremum is the capacity of the set.

PROOF: Let $E_1 \subset E_2 \subset \ldots$ be an increasing sequence of finite sets whose union is E. We must prove that if $\sup C(E_n)$ is finite, then h^E is a potential and $C(E) = \sup C(E_n)$. First we note that h^E is the monotone limit of h^{E_n}. If $i \in E_m$, then the ith component of e^{E_n} decreases for $n \geq m$. Thus $\lim e^{E_n} = \tilde{e}$ exists. Since $\hat{N}$ has columns that tend to zero and since $\eta^{E_n} \mathrm{II} = C(E_n) \leq \sup C(E_n) < \infty$,

$$\eta^{E_n} \hat{N} \to (\text{dual } \tilde{e})\hat{N}.$$

By duality,

$$h^{E_n} = Ne^{E_n} \to N\bar{e},$$

and

$\alpha\bar{e}$ = (dual $\bar{e}$) II = $\sum_i \lim \eta_i^{E_n}$, which by Fatou's theorem, is less than or equal to

$$\lim \eta^{E_n} II = \sup C(E_n).$$

Hence $h^E = N\bar{e}$ and $\alpha\bar{e} < \infty$, and so E is an equilibrium set. Also

$$C(E) = \alpha\bar{e} \leq \sup C(E_n).$$

But $C(E_n) \leq C(E)$ for every n, so that $\sup C(E_n) \leq C(E)$. Thus

$$C(E) = \sup C(E_n).$$

The converse, that an equilibrium set has the property that the supremum of the capacities of its finite subsets is finite, follows trivially from the monotonicity and the finiteness of capacity on equilibrium sets.

4.3. Extended chains and h-processes

Consider the state space S and two distinguished elements {a, b} added to it. Let Ω be the set of doubly infinite sequences $\omega = (\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots)$ such that

(i) $\omega_i \in S \cup \{a, b\}$

(ii) $\omega_i = a$ then $\omega_j = a \quad \forall\, j < i$ (state a stands for " not yet started ")

(iii) $\omega_i = b$ then $\omega_j = b \quad \forall\, j > i$ (state b stands for " stopped ")

(iv) $\omega_i \in S$ for at least one i (each path in Ω represents some non trivial possibility for the process).

For $i \in Z = \{0, \pm 1, \pm 2, \ldots\}$ define $X_i(\omega) = \omega_i$ and let $\mathcal{B} = \sigma(X_i, i \in Z)$.

An *extended chain* is determined by a measure μ on $(\Omega, \mathcal{B})$ such that

$\mu(\{\omega: X_i = j\}) < \infty \quad \forall\, j \in S$ and $\forall$ i.

For any set E in S, let

$$U_E(\omega) = \inf(n : X_n(\omega) \in E), \text{ possibly } -\infty$$

$$V_E(\omega) = \sup(n : X_n(\omega) \in E), \text{ possibly } +\infty$$

The domains of U_E and V_E are equal : $D(U_E) = D(V_E) = \{\omega : X_n(\omega) \in E$ for at least one $n\}$. When $E = S$, U_E and V_E are respectively the *initial* and the *final* times and are denoted by U and V.

Given E in S, define for all $n \geq 0$,

$$y_n^E(\omega) = X_{n+U_E(\omega)}(\omega) \qquad \text{if } U_E(\omega) > -\infty$$

Then y_n^E is a stochastic chain in the usual sense with state space $S \cup \{b\}$ and is called the *chain watched starting in* E. For finite E in S, let

$$\mu_i^{E,m} = \mu(U_E = m, X_m = i),$$

$$\mu^E = \sum_m \mu^{E,m}.$$

$(\mu^E)_{i \in E}$ is the *initial measure* of the process watched starting E.

Also let $\nu_j = \sum_n \mu(X_n = j)$. Then ν_j is the *expected number of visits to* j, that is, $\nu_j = (\mu^E N)_j$ for $j \in E$.

An extended chain is an *extended Markov chain* if for every finite set E in S

(i) $\mu(U_E = -\infty) = 0$.

(ii) y_n^E is a Markov chain with transition matrix P and finite starting measure.

(iii) for each $j \in S - \{a, b\}$, $\nu_j < \infty$, that is, P is transient.

THEOREM 4.9. Consider an extended chain with transition matrix P and let P' be the restriction of P to S. Then

(i) ν is a superregular measure for P'

(ii) $\nu(I - P') = \mu$ where $\mu = \lim \mu^E$ when $E \uparrow S$.

PROOF: $\nu^E P' = \mu^E N P' = \mu^E (N - I) = \nu^E - \mu^E$.

Since $\nu^{En} \uparrow \nu$ as $E_n \uparrow S$, the monotone convergence theorem implies that

$\nu P' = \nu - \lim \mu^{En}$ as $n \to \infty$, which establishes (i) and (ii)

For any $\omega \in \Omega$, define ω' to be the point in Ω with

$$X_n(\omega') = X_{-n}(\omega) \quad \text{if } X_{-n}(\omega) \in S$$
$$= a \quad \text{if } X_{-n}(\omega) = b$$
$$= b \quad \text{if } X_{-n}(\omega) = a$$

Let X_n be an extended chain on Ω with measure μ and set $\hat{P}(\omega \in A) = P(\omega' \in A)$ for all sets A for which the right hand side is defined. The extended stochastic chain defined by the measure $\hat{P}$ is called the *reverse* of the extended chain.

THEOREM 4. 10. If $\{X_n\}$ is an extended Markov chain, then its reverse is also an extended chain with the roles of U_E and V_E interchanged and transition matrix $\hat{P}$ with

$$\hat{P}_{ji} = P_{ij} \frac{v_i}{v_j} \quad \text{for all } i, j \in S.$$

For the proof see [1].

A useful transformation of transient chains into new transient chains in which row and column vectors are transformed into vectors of the same type is the following. Let h be a positive finite-valued superregular function for a transient chain P. The *h-process* is a Markov chain P^h with transition probabilities

$$P^h_{ij} = P_{ij} \frac{h_j}{h_i} .$$

The *h-process transformation* is a transformation defined on square matrices, measures and functions as follows

$$A^h_{ij} = A_{ij} \frac{h_j}{h_i}, \quad \mu_i{}^h = \mu_i h_i, \quad f_i{}^h = \frac{f_i}{h_i} .$$

It is easily seen that P^h is a transition matrix, that all states are transient, and that if the states of P communicate, the same is true for P^h. Moreover powers of P transform to the P^h process the same way that P does and

$$P^h N = N^h$$

where

$$P^h N = \sum_{n=0}^{\infty} (P^h)^n.$$

Also sums and products are preserved in the given order; and equalities, inequalities, and limits are preserved entry-by-entry. Any superregular function for P transforms into a superregular function for P^h. If α is the fixed reference superregular measure for P, then let α^h be the reference superregular measure for P^h.

If $h \geq 0$ is a finite-valued P-superregular function and π is the starting measure of the Markov chain with transition matrix P, $\pi h = 1$, then the h-chain is defined to be the Markov chain with state space S and with measure P^h defined by

$$P^h(X_0 = x_0, \ldots, X_n = x_n) = \pi_{x_0} P_{x_0,x_1} \cdots P_{x_{n-1},x_n} h(x_n).$$

If $S^h = \{i \in S,\ h_i > 0\}$, then the transition matrix P^h of the h-chain satisfies

$$P^h_{ij} = P_{ij} \frac{h_j}{h_i} \qquad \text{for } i, j \text{ in } S^h$$

$$= 0 \qquad \text{for } i \text{ in } S^h,\ j \text{ in } S - S^h$$

and the initial measure π^h satisfies

$$\pi^h_i = \pi_i h_i.$$

If $i \in S - S^h$, then P^h_{ij} is not defined and we set $P^h_{ib^*} = 1$.

Then

$$N^h_{ij} = N_{ij} \frac{h_j}{h_i} \qquad \text{if } i, j \in S^h$$

$$= \delta_{ij} \qquad \text{otherwise.}$$

4.4. Martin exit boundary

The Martin exit boundary allows us to describe the long-range behavior of the process and to obtain a Poisson-type representation for all finite-valued superregular functions which are integrable with respect to the starting distribution.

Let P be a transition matrix for a transient chain and let π be a starting vector such that $\pi \geq 0$ and $\pi \mathbb{1} = 1$. Assume that πN is positive.

Define

$$K(i, j) = \frac{N_{ij}}{(\pi N)_j}, \text{ where } (\pi N)_j = \sum_i \pi_i N_{ij}.$$

It is easily seen that for each j, $K(., j)$ is non-negative, finite-valued, everywhere regular except at j where it is strictly superregular, and $\pi K(., j) = 1$. For fixed i the function $K(i, .)$ is bounded, since

$$K(i, j) = \frac{N_{ij}}{(\pi N)_j} = \frac{1}{(\pi N)_i} \frac{(\pi N)_i N_{ij}}{(\pi N)_j} = \frac{1}{(\pi N)_i} (\text{dual } N_{ij}) \leq \frac{1}{(\pi N)_i} (\text{dual } N_{ii}) = \frac{N_{ii}}{(\pi N)_i}$$

where the duality is taken with respect to πN.

The set S is metrized by the distance function defined by

$$d(j, j') = \sum w_i (\pi N)_i |K(i, j) - K(i, j')|$$

where the w_i are positive weights such that $\sum w_i N_{ii}$ is finite.

A sequence $\{j_n\}$ in the metric space (S, d) is Cauchy if and only if the sequence of real numbers $\{K(i, j_n)\}$ is Cauchy for every i. To see this note that $w_i(\pi N)_i > 0$, hence if $\{j_n\}$ is Cauchy so is $\{K(i, j_n)\}$.

Conversely let $\{K(i, j_n)\}$ be Cauchy for all i, and let $\varepsilon > 0$ be given. Choose a finite set of states E such that

$$\sum w_k N_{kk} < \varepsilon / 2 \quad \text{where k ranges S - E}$$

and choose M so large that

$$|K(i, j_n) - K(i, j_m)| < \frac{\varepsilon}{2 \sum w_k N_{kk}}, \quad i \in E \text{ and } \forall\, n, m \geq M.$$

Then $d(j_m, j_n) < \varepsilon$, which shows that the sequence $\{j_n\}$ is Cauchy in (S, d).

Define S^* to be the Cauchy completion of (S, d) and let $B = S^* - S$. The set B is the *Martin exit boundary* for the chain P started with measure π.

The *Martin entrance boundary* is defined to be the Martin exit boundary for the reverse Markov chain.

The function $K(i, .)$ extends uniquely to a continuous function on S^* due to its uniform continuity on S. Also we know that $K(i, x_n)$ is bounded for any sequence $\{x_n\}$ of states and fixed i. Using a diagonal process argument we may choose a subsequence $\{x_{n_k}\}$ such that $\{K(i, x_n)\}$ is Cauchy for all i. Then $\{x_{n_k}\}$ is Cauchy in S^* which is complete, hence $\{x_{n_k}\}$ is convergent. In other words, S^* is compact.

The next theorem states that with probability one every path ω has the property that either $X_n(\omega)$ converges in S^* or the process along ω disappears in finite time.

THEOREM 4. 11. [1] Let the chain P with all states transient be started with a distribution π such that $\pi N > 0$. For each ω, let $V(\omega)$ be the last time in S. Then a.e. either $V(\omega) < \infty$ and $X_{V(\omega)}(\omega)$ is in S, or $V(\omega) = +\infty$ and $X_n(\omega)$ converges to a point in S^* as n goes to infinity.

COROLLARY. $P_i(X_V \in S^*) = 1$.

PROOF: By the previous theorem almost every path in the process P (started according to π) satisfies $x_v \in S^*$. Hence the same is true of the extended chain, and hence it is true of those paths in the extended chain which pass through i. On any path ω which passes through i, $x_v(\omega)$ is in S^* if and only if $x_v(\theta_{u(\omega)} \circ \omega)$ belongs to S^*. Therefore, the extended chain watched starting in $\{i\}$ satisfies

$$P_\mu(X_V \in S^*) = \mu^i\{i\}.$$

On the other hand

$$P_\mu(X_V \in S^*) = \mu^i\{i\} P_i(X_V \in S^*).$$

Since $\mu^i\{i\} \neq 0$, then $P_\mu(X_V \in S^*) = 1$.

4.5. Poisson - Martin - Representation

Harmonic measure μ is defined on the Borel sets E of S^* by $\mu(E) = P(X_V \in E)$. This measure describes the long run behavior of the Markov chain X, as $\mu(E)$ stands for the probability that, in the limit, the chain winds up in the set E.

The conditional probabilities $P_i(X_V \in E)$ are related to μ by the following formula, whose truth is established in [1],

$$P_i(X_V \in E) = \int_E K(i, x)d\mu(x).$$

From the above corollary we get at once

$$1 = P_i(X_V \in S^*) = \int_{S^*} K(i, x)d\mu(x) \qquad \forall\, i \in S$$

which is nothing but the representation theorem as it applies to the function $\mathbb{1}$. Now the notion of h-process will allow us to find the representation of a positive finite-valued P-superrregular function h.

THEOREM 4. 12. If $h > 0$ is a finite-valued P-superregular function such that $\pi h = 1$, then

$$h_i = \int_{S^*} K(i, x)d\mu^h(x)$$

PROOF: The representation of the function $\mathbb{1}$ using the h-process is given by

$$1 = \int_{S^*} K^h(i, x)d\mu^h(x)$$

or $$h_i = \int_{S^*} K(i, x)d\mu^h(x) \qquad \text{by definition of } K^h.$$

The representation theorem immediately extends to cover all finite-valued P-superregular functions $h \geq 0$ for which $0 < \pi h < \infty$.

To get uniqueness of the above representation we need another hypothesis.

PROPOSITION 4. 13. K(i, x) is normalized (i.e. $\pi K = 1$) for a.e. x [μ] in S*.

PROOF: We know that $\int_{S^*} K(i, x)d\mu(x) = 1$, so by Fubini's theorem we have

$$1 = \pi 1\!\!1 = \pi \int_{S^*} K(i, x)d\mu(x) = \int_{S^*} \pi K(i,x)d\mu(x)$$

and since

$$\int_{S^*} 1\, d\mu(x) = 1 \text{ and } 1-\pi K \geq 0, \text{ we get } \pi K(., x) = 1 \text{ a.e..}$$

PROPOSITION 4. 14. K(., x) is regular for a.e. x [μ] in the boundary B of S*.

PROOF: We know that $\int_B K(i, x)d\mu(x) = P_i(X_v \in B)$. But the event $(X_v \in B)$ is shift invariant, so by Theorem 4.3 the function $P_i(X_v \in B)$ is regular.

By Fubini's theorem

$$\int_B P\, K(., x)d\mu(x) = P \int_B K(., x)d\mu(x) = \int_B K(., x)d\mu(x)$$

Since $PK(., x) \leq K(., x)$, we have $PK(., x) = K(., x)$ for almost every $x \in B$.

A finite-valued function $h \geq 0$ is *minimal* if

(i) h is regular

(ii) whenever $0 \leq h' \leq h$ with h' regular, then $h' = c\, h$.

A point $x \in S^*$ is called an *extreme point* of S^* if the function $K(., x)$ is minimal and normalized .

The set of extreme points is denoted by B_e. Let $\overline{S} = S \cup B_e$.

THEOREM 4. 15. If $h \geq 0$ is a normalized minimal function such that

$$h_i = \int_{S^*} K(i, x) d\nu(x)$$

for a Borel measure ν such that $\nu(S^*) = 1$, then ν is concentrated at a single point and that point is extreme.

THEOREM 4. 16. If $h \geq 0$ is a normalized superregular function such that

$$h_i = \int_{\overline{S}} K(i, x) d\nu(x)$$

for some measure ν on S^* with $\nu(S^*) = \nu(\overline{S}) = 1$, then $\nu = \mu^h$.

In summary, we have the following theorem.

THEOREM 4. 17. The π-integrable non-negative P-superregular functions h stand in one-one correspondence with the non-negative finite measures ν on the Borel sets of $\overline{S}$, the correspondence of h to ν being

$$h_i = \int_{\overline{S}} K(i, x) d\nu(x),$$

and satisfying $\pi h = \nu(\overline{S})$. The unique representation $h = N f + r$ with r regular arises by decomposing the integral over $\overline{S}$ into a part over S and a part over B_e. If h is normalized, then the measure that corresponds to h is μ^h.

CHAPTER 5

CONSTRUCTION OF MARTIN BOUNDARIES

5.1. Background

Various boundaries for the state space of a discrete Markov chain were introduced by Feller [11], Doob [9], Hunt [13], and Watanabe [19].

The boundary introduced by Feller is purely analytic and no probabilistic argument or results are used. To illustrate the meaning of the boundary, let G be a simply connected domain of the Euclidean plane with a complicated boundary Γ. A conformal mapping reduces the theory of *bounded harmonic functions* in G to the theory of bounded harmonic functions in the circular disc. The inverse map from the disc into G induces for G an *ideal* boundary Γ^* which is not topologically equivalent to Γ, and which is more natural for the theory of harmonic functions. In several dimensions an appropriate boundary was defined by Martin [16]. The formal analogue for harmonic functions to Feller's boundary is not the Martin boundary but a larger one. Specializing to the unsymmetric one-dimensional random walk the Feller boundary consists of two points [11] corresponding to the harmonic functions II and $(\frac{q}{p})^i$, $-\infty < i < +\infty$ and i integer. For the symmetric random walk in at least 3 dimensions, Feller's boundary consists of one point corresponding to the only non-negative harmonic function II.

Doob's and Hunt's boundaries are essentially the same.

Watanabe's approach is motivated by Feller's work [11] and is somewhat different from Doob's. In establishing his boundary theory, the procedure is essentially the same as in Martin's original paper [16], but he introduces some modification to prove the main representation theorem. His boundary is not necessarily compact, differently from Martin's case.

H. Watanabe [18] and J. Feldman [20] studied some relations between the Martin boundary and the Feller boundary.

The version of the Martin boundary due to Doob was retained by most of probabilists and was described for various particular Markov chains.

In this thesis we follow a variation of Hunt's approach described in Kemeny, Snell and Knapp [1].

5.1.1. Random walks

Doob, Snell, and Williamson [8] gave the following theorems which characterize the minimal functions for random walks in N-dimensions.

THEOREM 5. 1. For a random walk in N-dimensions, the function 1l is minimal.

THEOREM 5. 2. For a random walk in N-dimensions a function h, positive with h(0)=1, is minimal if and only if it satisfies the following two conditions:

(a) $\sum h(i)p(i) = 1$

(b) $h(i+j) \doteq h(i)h(j)$ i, j in the state space S

As an application they calculated the Martin boundary of the one-dimensional p-q random walk : it reduces to two points corresponding to the two minimal functions 1l and $(\frac{q}{p})^i$.

Ney and Spitzer [17] gave a concrete realization of the Martin boundary for random walks in N-dimensions, N > 1 with transition function P(x, y) = P(0, x-y), mean vector $\mu = \sum xP(0, x) \neq 0$ and such that P(0, x) > 0 for only finitely many x and for some in each half space. Then the Green function $G(x, y) = \sum P^n(x, y)$ is finite and strictly positive. They used a result of Hennequin [12] according to which the function $\phi(a) = \sum P(0, x)e^{a.x}$, $a \in \mathbf{R}^N$ has the property that $D = \{a / \phi(a) \leq 1\}$

is a compact convex subset of $\mathbf{R}^N$, and grad(ϕ) is non-zero on the boundary ∂D of D.

Finally the sphere $S_{N-1} = \{x \in \mathbf{R}^N / |x| = 1\}$ is mapped homeomorphically onto ∂D by

$$\frac{\text{grad}\phi(u)}{|\text{grad}\phi(u)|} = v \rightarrow u = \alpha(v).$$

Then $\mathbb{Z}^N \cup S_{N-1}$ is a compact metric space with the metric

$$\begin{aligned} \rho(x, y) &= |x^* - y^*| && \text{if } x, y \in \mathbb{Z}^N \\ &= |x^* - y| && \text{if } x \in \mathbb{Z}^N,\ y \in S_{N-1} \\ &= |x - y| && \text{if } x, y \in S_{N-1} \end{aligned}$$

where $x^* = \frac{x}{1 + |x|}$ and in this topology the functions $f_y(.)$ defined for each $y \in \mathbb{Z}$ by

$$\begin{aligned} f_y(x) &= \frac{G(y, x)}{G(0, x)} && \text{if } x \in \mathbb{Z}^N \\ &= \exp(\alpha(x).y) && \text{if } x \in S_{N-1} \end{aligned}$$

are continuous on $\mathbb{Z}^N \cup S_{N-1}$. Thus S_{N-1} acts as the Martin boundary for this process.

The usual Poisson-Martin representation for the non-negative harmonic functions is given by

$$h(y) = \int_{S_{N-1}} \exp(\alpha(x).y)\, \mu(dx),$$

where μ is a probability measure on S_{N-1}

Lamperti and Snell [15] gave a set of results characterizing the minimal functions for space-time random walks.

5. 1. 2. Polya's urn scheme.

In 1923 Eggenberger and Polya introduced what is known as " the urn scheme" as a model for the development of a contagious phenomenon. A box contains b black balls and r red balls, and a ball is drawn from it at random with

'double replacement' (i.e. whatever ball is drawn it is returned to the box together with a fresh ball of the *same color*); the procedure is then continued indefinitely.

Blackwell and Kendall [7] calculated the Martin boundary for the case b = r = 1 and with an arbitrary number $k \geq 2$ of colors. For the Markov chain associated with the k-color urn scheme the Martin boundary ∂E is homeomorphic with the standard Euclidean (k-1)-cell, and for sample paths starting at (1,...,1) the distribution of the limiting point at the boundary is uniform over the whole of ∂E when this is identified with $U_{k-1} = \{x \in \mathbf{R}^k: x_i \geq 0, i = 1,..., k, \sum x_i = 1\}$.

5. 1. 3. Binary tree process

A binary tree process is a process starting at a point 0 which is the first node and then from each node there is a probability of 1/2 of going to each of the two succeeding nodes. The state space can be identified with the set of all finite sequences of zeroes and ones. For this process, Dawson [4] found the Martin boundary to be homeomorphic to the interval (0,1) with the natural topology, (the natural topology has been studied by A. Fabien [6]) in other words the boundary consists of the set of all infinite sequences of 0 and 1.

5. 1. 4. Other examples

Kemeny, Snell, and Knapp [1] have constructed Martin boundaries for the following Markov chains.

1. The renewal chain on the non-negative integers with transition probabilities

$$P_{i,i+1} = p_i, \quad P_{i,0} = q_i, \quad P_{0,1} = 1, \quad p_i + q_i = 1$$

has one boundary point corresponding to the regular function $\mathbb{1}$.

2. For the p-q random walk (p>q>0), looking at the function K(i, j) it is seen that the boundary consists of two points both extreme, and if the reference measure π is chosen in such a way that there is a P-regular function which is not π-integral, then

a boundary point whose function is identically 0 (hence it is not extreme) is introduced.

3. For the symmetric random walk in three dimensions it is seen that the only non-negative regular functions (bounded or not) are the constants.

4. The space-time coin tossing Markov chain which is defined on the set of lattice points (n,i) in the plane with $0 \le i \le n$ and probability transitions

$$P_{(n,i)(n+1,i)} = P_{(n,i)(n+1,i+1)} = \frac{1}{2}$$

has a boundary homeomorphic to the unit interval [0, 1] with the usual topology. Moreover all the points are extreme.

5. 1. 5. Branching processes

The Galton-Watson process (discrete, one-type branching process) is a Markov chain Z_n, $n \ge 0$, with the non-negative integers as state space. Its transition function

$$P(i, j) = P(Z_{n+1} = j / Z_n = i) = \sum_{k_1+\ldots+k_i=j} \Pi p_{k_i}$$

is defined in terms of a given probability measure $\{p_k\}$ on the non-negative integers. The generating function and its iterates are defined by

$$f_1(s) = f(s) = \sum p_k s^k, \qquad f_0(s) = s, \qquad f_{n+1}(s) = f(f_n(s)), \quad |s| \le 1$$

A full discussion of definitions and elementary properties is given in [2; chapter 1], which shows that , except in the trivial case $p_1 = 1$, the Green function

$$N_{ij} = \sum_{n\ge 0} P_{ij}^n \qquad \forall\, i \ge 0 \text{ and } \forall\, j \ge 1,$$

is finite.

Let m and σ^2 be respectively the mean and variance of $\{p_k\}$. Then we have a summary of results in the following chart.

Assumptions	Entrance boundary	Exit boundary
$m=1, \sigma^2<\infty$	one point [14]	one point [14]

m=1	one point [22]	one point [22]
m<1, E(klogk)<∞	the interval [0, 1) [21]	
m>1, E(klogk)<∞	the interval [0, 1) by reducing to the case m<1 [10] see below	

Let $m > 1$, then the probability of extinction satisfies $q < 1$. Let $h(i) = q^i$. Then $h(i)$ is a regular function. Form the h-process. The probability function of the corresponding h-process is given by $f^h(s) = \sum p_k^h s^k = \sum p_k q^k s^k = f(sq)$ and since $f^h(1) = f(q) = q<1$, then the p.g.f is defined as $\frac{f(sq)}{q}$. We have $\left(\frac{f(sq)}{q}\right)_{s=1} = f'(q) < 1$.

Hence the h-process is a subcritical Galton-Watson process and its entrance boundary is the same as the entrance boundary of the original process i.e. the interval [0, 1) [21].

For the Galton-Watson process with immigration described by ([25], [26]), Pakes ([23], [24]) has given conditions under which the Markov chain is transient and studied the asymptotic properties of the Green function $N_{ij} = \sum P_{ij}^n$ $(i, j = 0, 1, ...)$. A unique limit to $\frac{N_{ij}}{N_{i0}}$ is obtained when i goes to ∞ which shows that the entrance boundary consists of one point.

5.2. Explicit construction of martin boundaries

The following proposition will be useful in the sequel.

PROPOSITION 5.3. For any i, j, k in the state space of a Markov chain we have

$$H_{ij} \geq H_{ik}H_{kj}$$

and the equality holds if any transition from i to j has to pass through k before j.

PROOF: Let τ_j be the time of the first visit to j, then by the strong Markov property

$$H_{ij} = P_i(\tau_j < \infty) \geq P_i(\tau_k < \infty, \tau_k \leq \tau_j < \infty).$$

Now $P_i(\tau_k < \infty, \tau_k \le \tau_j < \infty) = \int 1_{\{\tau_k < \infty\}} 1_{\{\tau_k \le \tau_j < \infty\}} d P_i$

$$= \int 1_{\{\tau_k < \infty\}} E_i[1_{\{\tau_j \circ \theta\tau_k < \infty\}} / \mathcal{B}_{\tau_k}] d P_i$$

$$= \int 1_{\{\tau_k < \infty\}} P_k [\tau_j < \infty] d P_i$$

$$= H_{ik} H_{kj}.$$

Example 5.2.1. Birth-and-death chain

5.2.1.1. The Boundary

Consider a birth-and-death chain on the non-negative integers with transition probabilities

$$P_{(i,i+1)} = p_i, \quad P_{(i,i-1)} = q_i, \quad p_0 = 1, \quad p_i + q_i = 1, \text{ with } p_i > 0, \ q_i > 0.$$

It is known [4] that this chain is transient if and only if $\sum_{j=1}^{\infty} \Pi_{k=1,\dots,j} \left(\frac{q_k}{p_k} \right) < \infty$ which will be assumed in the sequel. Since every state can be visited from the state 0, let π have all its weight at 0. Then

$$K(i, j) = \frac{N_{ij}}{N_{0j}} = \frac{H_{ij} N_{jj}}{H_{0j} N_{jj}} = \frac{H_{ij}}{H_{0j}}, \qquad \text{for } i \le j.$$

Since the chain is transient $H_{ij} = H_{0j} = 1$ for $i \le j$ and by Proposition 5.3

$$H_{0j} = H_{01} H_{12} \dots H_{j-1,j},$$

hence

$$K(i, j) = \frac{H_{ij}}{H_{0j}} = 1.$$

Therefore,

$\lim K(i, j_n) = 1 = K(i, \infty)$ independently of n and the whole state space $\{1, 2, \dots\}$ forms one Cauchy sequence with respect to the topology induced by the distance

$$d(j, j') = \sum \omega_i N_{0i} \, | K(i, j) - K(i, j') |.$$

Since there is only one boundary point it must be extreme. Moreover since H is regular every regular function must be a constant. Since the process does not disappear harmonic measure assigns unit mass to $+\infty$.

A quick generalization can be obtained by considering what may be called a two-sided birth-and-death process which is obtained by connecting together at 0 two similar birth and death processes of the above type. Let the state space S be the union of the two state spaces S_1 and S_2. Then starting at 0 the probability is 1/2 to proceed along either one.

Taking the limit for j in S_1,

$$K(i, \infty_1) = \lim_j \frac{H_{ij}}{H_{0j}} = \lim_j \frac{H_{ij}}{H_{0i}H_{ij}} = \frac{1}{H_{0i}} \qquad \text{if } i \in S_1$$

$$= \lim_j \frac{H_{i0}H_{0j}}{H_{0j}} = H_{i0} \qquad \text{if } i \in S_2$$

and similarily,

$$K(i, \infty_2) = H_{i0} \qquad \text{if } i \in S_1$$

$$= \frac{1}{H_{0i}} \qquad \text{if } i \in S_2.$$

Since $\lim K(i, \infty_1) = \frac{1}{H_{0\infty_1}} \neq \lim K(i, \infty_2) = 0$, where the limit is taken over $i \in S_1$, then $K(i, \infty_1)$ and $K(i, \infty_2)$ are two distinct functions which are easily seen to be harmonic, normalized and minimal. Hence there are two boundary points both extreme. Harmonic measure μ puts all its mass on $\{\infty_1, \infty_2\}$ and by symmetry we have $\mu(\{\infty_1\}) = \mu(\{\infty_2\}) = \frac{1}{2}$.

To summarize we have :

THEOREM 5. 4. The Martin boundary of k copies ($1 \leq k < \infty$) of a birth-and-death Markov chain defined on the non-negative integers contains k points, all extreme. The relative topology for S is the discrete topology.

5.2.1.2. Potentials

Let us return to the simple birth-and-death chain. In P if $g=Nf$ is a potential, then

$$g_i=(Nf)_i=\sum N_{ik}f_k,$$

where

$$N_{ij}=H_{ij}N_{jj},\quad N_{jj}=\frac{1}{1-\underline{H}_{jj}}\quad\text{and}\quad \underline{H}_{ij}=p_jH_{j+1,j}+(1-p_j)H_{j-1,j}\quad\text{for } j\geq 1.$$

Since $H_{ij}=1$ for $i\leq j$ we calculate

$$N_{ij}=\frac{1}{p_j(1-H_{j+1,j})}\qquad \text{if } i\leq j$$

$$=\frac{H_{ij}}{p_j(1-H_{j+1,j})}\qquad \text{if } i>j$$

$$g_i=\sum_{k=0}^{i-1}\frac{H_{ik}}{p_k(1-H_{k+1,k})}f_k+\sum_{k=i}^{\infty}\frac{1}{p_k(1-H_{k+1,k})}f_k=N_i.f.$$

Note that g is finite-valued if $N_i.|f|<\infty$ for every row N_i of N.

Clearly the birth-and-death process will visit any infinite set infinitely often a.s.. Hence only finite sets are equilibrium sets. Let us verify this fact in terms of equilibrium potentials.

Let $E=\{m, m+1, ..., m+n\}$ be a finite set, then its potential is given by the hitting vector

$$h_i^E=P_i(\text{process eventually reaches } E),$$

which for the birth-and-death process is,

$$h_i^E=1\qquad \text{if } i\leq n+m$$

$$=H_{i,n+m}\qquad \text{if } i>n+m.$$

Also, $h^E=Ne^E$, where e^E is the escape vector given by

$e_i^E=P_i\{$process goes on first step from E to S-E and then never return to E$\}$, which for our process looks like

$$e_i^E=p_{n+m}(1-H_{m+n+1,m+n})\qquad \text{if } i=n+m$$

$$=0\qquad \text{otherwise.}$$

Let the reference measure be N_r, where N_r is some row of the matrix N. The capacity of E, if we assume that $n+m\geq r$, is given by

$$C(E)=N_re^E=N_{r,n+m}e_{n+m}^E=\frac{p_{n+m}(1-H_{m+n+1,m+n})}{p_{n+m}(1-H_{m+n+1,m+n})}=1.$$

If the set E is such that $r > n+m$, its capacity is given by-

$$C(E)=N_r e^E = N_{r,n+m}e^E_{n+m}= \frac{H_{r,n+m}\, p_{n+m}\,(1- H_{m+n+1,\, m+n})}{p_{n+m}\,(1- H_{m+n+1,\, m+n})} = H_{r,n+m}$$

If E is an infinite set, $h^E = \mathbb{1}$ and $e^E = (I-P)h^E = 0$. Hence h^E is not an equilibrium potential and E is not an equilibrium set.

For the double-birth-and-death process the situation is slightly different because from any finite set there are two possible routes for escape. Let $E = \{0 < m, m+1, ..., m+n\}$, then

$$e^E_i = p_{n+m}(1- H_{m+n+1,m+n}) \qquad \text{if } i = n+m$$

$$= q_m(1- H_{m-1,m}) \qquad \text{if } i = m$$

$$= 0 \qquad \text{otherwise}$$

and

$$h^E_i = H_{i,n+m} \qquad \text{if } i \geq n+m$$

$$= H_{i,m} \qquad \text{if } i \leq m$$

$$= 1 \qquad \text{otherwise.}$$

The capacity $C(E) = N_r e^E$

$$= N_{r,n+m}e^E_{n+m} + N_{r,m}e^E_m$$

$$= \frac{H_{r,n+m}\, p_{n+m}(1 - H_{m+n+1,m+n})}{1 - p_{n+m}H_{n+m+1,n+m} - q_{n+m}H_{n+m-1,n+m}} + \frac{H_{r,m}q_m(1 - H_{m-1,m})}{1 - p_mH_{m+1,m} - q_mH_{m-1,m}}$$

If the set E contains the state 0, $E = \{m', ...0, ..., m\}$, then

$$e^E_i = p_m(1 - H_{m+1,\, m}) \qquad \text{if } i = m$$

$$= p_{m'}(1 - H_{(m+1)',\, m'}) \qquad \text{if } i = m'$$

$$= 0 \qquad \text{otherwise,}$$

$$C(E) = \frac{p_m(1 - H_{m+1,m})}{1 - p_mH_{m+1,m} - q_mH_{m-1,m}} + \frac{p_{m'}(1-H_{(m+1)',m'})}{1 - p_{m'}H_{(m+1)',m'} - q_{m'}H_{(m-1)',m'}}$$

S is not an equilibrium set because $\hat{N}$ does not have columns tending to zero:

$$\hat{N}_{kj} = \frac{N_{ij}}{N_{ik}} N_{jk} = \frac{N_{ij}}{N_{ik}} H_{jk} N_{kk} = \frac{N_{ij}}{N_{ik}} \frac{H_{jk}}{1-\underline{H}_{kk}}$$

Substituting for $N_{ik} = \frac{H_{ik}}{1-\underline{H}_{kk}}$, $N_{ij} = \frac{H_{ij}}{1-\underline{H}_{jj}}$

$\hat{N}_{kj} = \frac{H_{ji}H_{jk}}{(1-H_{jj})H_{ki}}$ which does not go to zero when $k \to \infty$.

5.2.2. The space-time random walk

Consider the following Markov chain defined on $\mathbb{N} \times \mathbb{Z}$ and with transition probabilities

$$P_{(n,i)(n+1,i+1)} = p \qquad n \geq 0, i \in \mathbb{Z}.$$

$$P_{(n,i)(n+1,i-1)} = q \qquad p + q = 1.$$

Since there are exactly $\frac{j_1!}{((j_1+j_2)/2)!((j_1-j_2)/2)!}$ different paths from $(0, 0)$ to (j_1, j_2) [5], then

$$H_{(0,0)(j_1,j_2)} = \frac{j_1!}{((j_1+j_2)/2)!((j_1-j_2)/2)!} p^{(j_1+j_2)/2} q^{(j_1-j_2)/2}$$

and

$$H_{(i_1,i_2)(j_1,j_2)} = H_{(0,0)(j_1-i_1,j_2-i_2)} = \frac{(j_1-i_1)!}{((j_1+j_2-i_1-i_2)/2)!((j_1-j_2-i_1+i_2)/2)!} p^{(j_1+j_2-i_1-i_2)/2} q^{(j_1-j_2-i_1+i_2)/2}$$

for $j_1 \geq i_1$, $|j_2| \leq j_1$ and $|i_2| \leq i_1$, also $j_1 + j_2$ and $i_1 + i_2$ must be even numbers.

Hence

$$K[(i_1, i_2), (j_1, j_2)] = \frac{H_{(i_1,i_2)(j_1,j_2)}}{H_{(0,0)(j_1,j_2)}} =$$

$$= \frac{(j_1+j_2)(j_1+j_2-2)\ldots(j_1+j_2-i_1-i_2+2)(j_1-j_2)(j_1-j_2-2)\ldots(j_1-j_2-i_1+i_2+2)}{j_1(j_1-1)\ldots(j_1-i_1+1)} \left(\frac{1}{2}\right)^{i_1} p^{(-i_1-i_2)/2} q^{(-i_1+i_2)/2}$$

Noting that both numerator and denominator have exactly the same number of factors containing j_1, then dividing every factor by j_1 we obtain an expression which is a function of $\frac{j_2}{j_1}$.

Hence if (j_1, j_2) goes to ∞ in such a way that $\frac{j_2}{j_1}$ converges to some $t \in [0, 1]$, then the limit of $K((i_1, i_2), (j_1, j_2))$ exists and is given by

$$K((i_1, i_2), t) = \left(\frac{1}{2}\right)^{j_1} \left(\frac{1+t}{p}\right)^{(i_1+i_2)/2} \left(\frac{1-t}{q}\right)^{(i_1-i_2)/2}$$

Hence whenever $\lim \frac{j_2}{j_1}$ exists in $[0, 1]$, the sequence $\{j_1, j_2\}$ is Cauchy in S^*. Conversely, if $\{j_1, j_2\}$ is Cauchy in S^*, then $\lim \frac{j_2}{j_1}$ exists. In fact set $i_1 = 1$, $i_2 = 1$ and consider

$$K((1, 1), (j_1, j_2)) = \frac{H_{(1,1)(j_1 j_2)}}{H_{(0,0)(j_1 j_2)}} = p^{-1}\left(1 + \frac{j_2}{j_1}\right)$$

But by assumption $\{j_1, j_2\}$, and hence $K(., (j_1, j_2))$, is Cauchy which implies that $\{\frac{j_2}{j_1}\}$ is convergent.

Therefore the class of infinite Cauchy sequences is in one-to-one correspondence with the rays to infinity, that is, with points in the interval $[0, 1]$. The functions associated with these sequences are $K((i_1, i_2), t)$.

Now let us check that these functions are harmonic, normalized, and minimal.

Since there are only two possible one-step transitions harmonicity is easily checked.

Hence $K((i_1, i_2), t)$ is regular and the interval $[0, 1]$ forms the entire boundary.

Also, the reference measure π being concentrated at (0,0) we get at once

$$\pi K(., t) = 1K((0,0), t) = 1,$$

which proves that $K((i_1, i_2), t)$ are normalized.

To see that every point of the boundary is extreme, suppose that for some t_0, the function $K(., t_0)$ is not minimal. By the representation theorem 4.17, there exists a measure ν on the Borel sets of $B_e \subset [0, 1]$ with

$$K((i_1, i_2), t_0) = \int_{B_e} K((i_1, i_2), t) d\nu(t).$$

Now setting $i_1 = i_2$ and extend ν to $[0, 1]$ we have after simplification:

$$(1+t_0)^{i_1} = \int_{[0,1]} (1+t)^{i_1} d\nu(t) \quad \forall\, i_1.$$

Hence ν is concentrated at t_0 and t_0 must be extreme.

In summary we have:

THEOREM 5. 5. The Martin exit boundary of the space-time random walk process with transition probabilities p and q (p+q=1) is homeomorphic to the unit interval [0,1]; the homeomorphism being given by

$$K((i_1, i_2),t) = \left(\frac{1}{2}\right)^{i_1} \left(\frac{1+t}{p}\right)^{(i_1+i_2)/2} \left(\frac{1-t}{q}\right)^{(i_1-i_2)/2}.$$

Moreover every point of the boundary is extreme.

A somehow similar situation is given by the following process.

Let S be the set of lattice points $(n, i_1, i_2, \ldots, i_{r-1})$ in $\mathbb{R}^r$ with $0 \le i_k$ and $i_1 + \ldots + i_{r-1} = 1$. A point of S is to be identified with i_1 occurrences of outcome1, i_2 occurrences of outcome 2, ... , i_{r-1} occurences of outcome r-1, of some experiment with r-1 equiprobable outcomes.

THEOREM 5. 6. The Martin boundary of the above r-dimensional random walk is homeomorphic to the unit square $[0, 1]^{r-1}$, the homeomorphism being given by

$$K((m, i_1, i_2, \ldots, i_{r-1}),(t_1, \ldots, t_{r-1})) = r^m \prod_{k=1}^{r-1} t_k^{i_k} \left(1 - \sum_{k=1}^{r-1} t_k\right)^{m - \sum i_k}.$$

Moreover every point of the boundary is extreme.

The proof is similar to the one given to theorem 5.5 and is skipped.

The strong law of large numbers, when applied to the above experiment, is equivalent with the statement that harmonic measure ν concentrates all its mass at the point t_0 whose coordinates are all equal to $\frac{1}{r}$. We can verify this result directly:

$$\int_{[0,1]^{r-1}} r^m \prod_{k=1}^{r-1} t_k^{i_k} \left(1-\sum_{k=1}^{r-1} t_k\right)^{m-\sum i_k} d\delta t_0 = r^m \left(\frac{1}{r}\right)^{\sum i_k} \left(\frac{1}{r}\right)^{m-\sum i_k} = 1,$$

which is the representation theorem applied to the regular function 1l. By uniqueness, δt_0 must be the harmonic measure for the process.

5.2.3 The tree process with infinite branching

This is a process starting at a point 0 which is the first node and then from each node there is a probability of p_i, $i \in \mathbf{N}$, of going to each of the countably infinite succeeding nodes. The state space can be identified with the set of all finite sequences of non-negative integers. Since each state is entered at most once , N=H. The hitting probability from $i=(0, i_1, \dots, i_n)$ to a distinct state $j=(0, i_1, \dots, i_n, i_{n+1}, \dots, i_{n+k})$ is $p_{i_{n+1}}p_{i_{n+2}}\cdots p_{i_{n+k}}$ and 0 if the initial segment of j is different from i.

Note that $K(i, j) = \frac{H_{ij}}{H_{0j}} = \frac{H_{ij}}{H_{0i}H_{ij}} = \frac{1}{H_{0i}}$ if i is an initial segment of j, and

$K(i, j) = 0$ otherwise.

Clearly a sequence of states $\{j_n\}$ is Cauchy if and only if each j_k is the initial segment of j_{k+1}. Hence $\{j_n\}$ converges to a point u in S^* where u is an infinite sequence of non-negative integers. This gives us the following theorem.

THEOREM 5. 7. The martin boundary of the above tree process coincides with $\mathbf{N}^{\mathbf{N}}$.

5.2.4. The double-renewal chain.

5.2.4.1. The boundary

Let the state space S be the union of $S_1=\{0, 1, \dots\}$ and $S_2=\{0', 1', \dots\}$.

The non-zero entries of the transition matrix P are

$$P_{i-1,i} = p_i\,; \qquad P_{i-1,0'} = 1 - p_i\,; \qquad P_{(i-1)',i'} = p_{i'}\,; \quad P_{(i-1)',0} = 1 - p_{i'},\ i \geq 1,\ i' \geq 1.$$

Let

$$\beta_0 = \beta'_{0'} = 1, \quad \beta_i = \Pi_{k=1}^{i} p_k, \quad \beta'_{i'} = \Pi_{k=1}^{i} p_{k'}, \quad \beta = \lim \beta_i, \quad \beta' = \lim \beta'_{i'}.$$

Assume that $\beta > 0$ and $\beta' > 0$.

PROPOSITION 5. 8. For $j \in S_1$, we have

$$H_{0j} = \beta_j + (1 - \beta_j)(1 - \beta')\, H_{0j} \quad \text{or}$$

$$H_{0j} = \frac{\beta_j}{1 - (1 - \beta_j)(1 - \beta')}$$

PROOF: The proof is based on the strong Markov property

$$E_0(Y \circ \theta_\tau / \mathcal{B}_\tau) = E_{X_\tau}(Y) \qquad P_0 \text{ a.e on } \{\tau < \infty\}.$$

Let τ_j be the time of the first visit to j, then

$$H_{0j} = P_0(\tau_j < \infty) = P_0(\tau_j < \tau_0, \tau_j < \infty) + P_0(\tau_0 < \tau_j, \tau_j < \infty)$$

$$= \beta_j + P_0(\tau_0 < \tau_j, \tau_j < \infty).$$

$$P_0(\tau_0 < \tau_j, \tau_j < \infty) = \int 1_{\{\tau_0 < \tau_j\}}\, 1_{\{\tau_j < \infty\}}\, dP_0$$

$$= \int 1_{\{\tau_0 < \tau_j\}}\, 1_{\{\tau_j < \infty\}} 1_{\{\tau_0 < \infty\}}\, dP_0$$

$$= \int 1_{\{\tau_0 < \tau_j\}}\, 1_{\{\tau_j \circ \theta_{\tau_0} < \infty\}} 1_{\{\tau_0 < \infty\}}\, dP_0$$

$$= \int 1_{\{\tau_0 < \tau_j\}}\, E_0[1_{\{\tau_j \circ \theta_{\tau_0} < \infty\}} / \mathcal{B}_{\tau_0}] 1_{\{\tau_0 < \infty\}}\, dP_0$$

$$= \int 1_{\{\tau_0 < \tau_j\}}\, E_{X_{\tau_0}}[1_{\{\tau_j < \infty\}}]\, 1_{\{\tau_0 < \infty\}} dP_0$$

$$= H_{0j}\, P_0(\tau_0 < \tau_j, \tau_0 < \infty)$$

$$P_0(\tau_0 < \tau_j, \tau_0 < \infty) = \int 1_{\{\tau_0 < \tau_j\}}\, 1_{\{\tau_0 < \infty\}}\, dP_0$$

$$= \int 1_{\{\tau_0 < \tau_j\}}\, 1_{\{\tau_{0'} < \tau_0\}} 1_{\{\tau_{0'} < \infty\}} 1_{\{\tau_0 < \infty\}}\, dP_0$$

$$= \int 1_{\{\tau_{0'} \leq \tau_j\}}\, E_0[1_{\{\tau_0 \circ \theta_{\tau_{0'}} < \infty\}} / \mathcal{B}_{\tau_{0'}}] 1_{\{\tau_{0'} < \infty\}}\, dP_0$$

$$= \int 1_{\{\tau_{0'} < \tau_j\}} E_{X_{\tau_{0'}}}[1_{\{\tau_0 < \infty\}}] 1_{\{\tau_{0'} < \infty\}} \, dP_0$$

$$= \int 1_{\{\tau_{0'} < \tau_j\}} P_{0'}(\tau_0 < \infty) 1_{\{\tau_{0'} < \infty\}} \, dP_0$$

$$= P_{0'}(\tau_0 < \infty) \, P_0(\tau_{0'} < \tau_j, \tau_{0'} < \infty) = (1-\beta')(1-\beta_j)$$

Hence the result.

Similarly for i and j in S_1, $0 < i < j$,

$$H_{ij} = \frac{\beta_j}{\beta_i} + P_i(\tau_i < \tau_j, \tau_j < \infty) = \frac{\beta_j}{\beta_i} + \left(1 - \frac{\beta_j}{\beta_i}\right)(1 - \beta')H_{0j}$$

THEOREM 5. 9. The 'double renewal chain' has two Martin exit boundary points both extreme.

PROOF: For $j \in S_1$ we have

$$K(i, j) = \frac{H_{ij}}{H_{0j}} = \frac{H_{i0}H_{0j}}{H_{0j}} = H_{i0} = \left(1 - \frac{\beta}{\beta_i}\right)(1 - \beta') \qquad \text{for } i \in S_1, j \le i$$

$$= \left(\frac{1 - (1- \beta_j)(1 - \beta')}{\beta_i} + \left(1 - \frac{\beta_j}{\beta_i}\right)(1 - \beta')\right) = 1 - \beta' + \frac{\beta'}{\beta_i} \qquad \text{for } i \in S_1, j > i$$

$$K(i', j) = \frac{H_{i'0}H_{0j}}{H_{0j}} = H_{i'0} = 1 - \frac{\beta'}{\beta_{i'}} \qquad \text{for } i' \in S_2.$$

Since $K(., j_n)$ is independent of $j_n \in S_1$ (for large enough j), we have at once a Cauchy sequence, namely S_1, with limit denoted by ∞. Corresponding to this limit is the function

$$K(., \infty) = \mathrm{Lim}_{j \in S_1, j \to \infty} K(., j)$$

which, we are going to show, is regular.

For $i \in S_1$

$$\sum_{k \in S} P_{ik}K(k,\infty) = P_{(i,i+1)}K(i+1, \infty) + P_{(i,0')}K(0', \infty)$$

$$= p_{i+1}\left(1 - \beta' + \frac{\beta'}{\beta_{i+1}}\right) + (1-p_{i+1})(1-\beta')$$

$$= p_{i+1} - p_{i+1}\beta' + \frac{\beta'}{\beta_i} + 1 - \beta' - p_{i+1} + p_{i+1}\beta' = 1 - \beta' + \frac{\beta'}{\beta_i}$$

$$= K(i, \infty)$$

Similarly for $i' \in S_2$

$$\sum_{k \in S} P_{i'k} K(k, \infty) = K(i', \infty).$$

For $j' \in S_2$ we have

$$K(i, j') = \frac{H_{ij'}}{H_{0j'}} = \frac{H_{i0'}H_{0'j'}}{H_{00'}H_{0'j'}} = \frac{H_{i0'}}{H_{00'}} = \frac{\beta_i - \beta}{\beta_i(1-\beta)} \qquad \text{for } i \in S_1.$$

$$K(i', j') = \frac{H_{i'j'}}{H_{0j'}} = \frac{H_{i'j'}}{H_{0i'}H_{i'j'}} = \frac{1}{H_{0i'}} = \frac{1}{(1-\beta)H_{0'i'}} = 1 + \frac{\beta}{\beta'_{i'}}(1-\beta) \quad \text{for } i' \in S_2, i' < j'$$

$$= \left(1 - \frac{\beta'}{\beta'_{i'}}\right) \qquad \text{for } i' \in S_2, i' \geq j'.$$

This allows us to say, by the same argument as for S_1, that S_2 is also a Cauchy sequence with limit denoted ∞' and we have $K(., \infty') \equiv K(., j')$.

Now

$$\text{Lim}_{i \in S_1 \text{ and } i \to \infty} K(i, \infty) = \text{Lim}_{i \in S_1 \text{ and } i \to \infty} \left(1 - \beta' + \frac{\beta'}{\beta_i}\right) = 1 - \beta' + \frac{\beta'}{\beta}$$

$$\neq \text{Lim}_{i \in S_1 \text{ and } i \to \infty} K(i, \infty') = 0.$$

This shows that ∞ and ∞' are distinct, hence the boundary consists of two points (∞, ∞'), both extreme.

Let us now find the harmonic measure μ. Consider the representation of the function H, which gives

$$1 = \int_{S^*} K(i, x)d\mu(x) = K(i, \infty)\mu(\infty) + K(i, \infty')\mu(\infty')$$

$$= \frac{\beta_i(1-\beta')+\beta'}{\beta_i}\mu(\infty) + \frac{\beta_i - \beta}{\beta_i(1-\beta)}\mu(\infty') \qquad \text{if } i \in S_1$$

$$= \frac{\beta'_{i'} - \beta'}{\beta'_{i'}}\mu(\infty) + \frac{\beta_{i'}(1-\beta)+\beta}{\beta_{i'}(1-\beta)}\mu(\infty') \qquad \text{if } i' \in S_2.$$

Letting (i, i') go to (∞, ∞') we obtain,

$$\mu(\infty) = \frac{\beta}{\beta - \beta\beta' + \beta'}$$

$$\mu(\infty') = \frac{\beta'(1-\beta)}{\beta-\beta\beta'+\beta'} .$$

5.2.4.2. Potentials

Next we find the general form of potentials. For the double renewal chain, if g=Nf, then

$$g_i = (Nf)_i = \sum_{j=0}^{\infty} N_{ij}f_j + \sum_{j'=0'}^{\infty'} N_{ij'}f_{j'}$$

In section 3.2 we saw that $N_{ij} = H_{ij}N_{jj}$, $N_{jj} = \frac{1}{1-\underline{H}_{jj}}$ for $j \in S_i$

For $j \in S_1$ we get, $\underline{H}_{jj} = p_{j+1}H_{j+1,0'}H_{0'j} + (1 - p_{j+1})H_{0'j}$

$$= H_{0'j}\left(p_{j+1}\left(1-\frac{\beta}{\beta_{j+1}}\right) + (1 - p_{j+1})\right)$$

$$= H_{0'j}\left(1-\frac{\beta}{\beta_j}\right).$$

But, $H_{0'j} = H_{0'0}H_{0j} = (1-\beta')H_{0j} = \frac{(1-\beta')\beta_j}{\beta'+\beta_j-\beta'\beta_j}$,

hence,

$$\underline{H}_{jj} = \frac{(1-\beta')(\beta_j-\beta)}{\beta'+\beta_j-\beta'\beta_j} \quad \text{and} \quad N_{jj} = \frac{1}{1-\underline{H}_{jj}} = \frac{\beta'+\beta_j-\beta'\beta_j}{\beta'+\beta-\beta'\beta} .$$

Now, $H_{ij} = H_{i0'}H_{0'j} = \left(1-\frac{\beta}{\beta_i}\right)\frac{(1-\beta')\beta_j}{\beta'+\beta_j-\beta'\beta_j}$ if $i \geq j \in S_1$

$$= \frac{\beta_j}{\beta_i} + \left(1-\frac{\beta_j}{\beta_i}\right)\frac{(1-\beta')\beta_j}{\beta'+\beta_j-\beta'\beta_j} = \frac{\beta_j}{\beta_i}\left(\frac{\beta'+\beta_i-\beta'\beta_i}{\beta'+\beta_j-\beta'\beta_j}\right) \quad \text{if } i < j \in S_1$$

$$H_{ij'} = H_{i0'}H_{0'j'} = \left(1-\frac{\beta}{\beta_i}\right)H_{0'j'} = \left(1-\frac{\beta}{\beta_i}\right)\frac{\beta'_{j'}}{\beta+\beta'_{j'}-\beta\beta'_{j'}} \quad \text{if } j' \in S_2 \text{ and } i \in S_1$$

$$N_{ij} = H_{ij}N_{jj} = \frac{\beta_j(1-\beta')(\beta_i-\beta)}{\beta_i(\beta'+\beta-\beta'\beta)} \quad \text{if } i \geq j \in S_1$$

$$= \frac{\beta_j(\beta'+\beta_i-\beta'\beta_i)}{\beta_i(\beta'+\beta-\beta'\beta)} \quad \text{if } i < j \in S_1$$

$$N_{ij'} = H_{ij'}N_{j'j'} = \left(1-\frac{\beta}{\beta_i}\right)\frac{\beta'_{j'}}{(\beta'+\beta-\beta'\beta)} \quad \text{if } j' \in S_2 \text{ and } i \in S_1.$$

Hence, for $i \in S_1$,

$$g_i = (Nf)_i = \sum_{j=0}^{\infty} N_{ij} f_j + \sum_{j'=0'}^{\infty'} N_{ij'} f_{j'}$$

$$= \sum_{j=i+1}^{\infty} \frac{\beta_j(\beta+\beta_i-\beta\beta_i)}{\beta_i(\beta'+\beta-\beta'\beta)} f_j + \sum_{j=0}^{i} \frac{\beta_j(1-\beta)(\beta_i-\beta)}{\beta_i(\beta'+\beta-\beta'\beta)} f_j + \sum_{j'=0'}^{\infty'} \left(1-\frac{\beta}{\beta_i}\right) \frac{\beta_{j'}}{(\beta'+\beta-\beta'\beta)} f_{j'}.$$

A similar expression holds for $g_{i'}$, where $i' \in S_2$.

Next, we determine some equilibrium sets for P. It is clear that the chain will be in any finite set only finitely often, hence all finite sets are equilibrium sets.

Let $E = \{k, k+1, ..., k+m\}$ be in S_1, then the escape vector of E is given by

$$e_i^E = (1-p_{i+1})(1-H_{0'k}) = (1-p_{i+1})(1-H_{0'0}H_{0k}) = (1-p_{i+1})\frac{\beta'}{\beta'+\beta_k-\beta'\beta_k} \qquad 0<k\le i<k+m$$

$$= \frac{\beta}{\beta_{k+m}} + (1-p_{k+m+1})\frac{\beta'}{\beta'+\beta_k-\beta'\beta_k} \qquad i=k+m$$

$$= 0 \qquad \text{otherwise.}$$

If the reference measure α is defined on $S_1 \cup S_2$ by (hence α is positive and superregular)

$$\alpha_j = \beta_j \qquad j \in S_1$$

$$\alpha_{j'} = \beta_{j'} \qquad j' \in S_2$$

the capacity of E is given by

$$C(E) = \alpha e^E = \frac{\beta'}{\beta'+\beta_k-\beta'\beta_k} \sum_{i=k}^{k+m-1} (1-p_{i+1})\beta_i + \beta_{k+m} e^E_{k+m}$$

$$= \frac{\beta'}{\beta'+\beta_k-\beta'\beta_k} \sum_{i=k}^{k+m} (\beta_i-\beta_{i+1}) + \beta.$$

As $k \to \infty$ $C(E) \to \beta$ and it is seen that β is the capacity of all finite sets in the one-renewal chain in other words the effect of S_2 is diminishing as states far away from 0 are considered.

The unique equilibrium potential with charge e^E is given by the hitting vector

$h^E = Ne^E$, or

$$h_i^E = \sum_{j=i}^{\infty} \frac{\beta_j(\beta'+\beta_i-\beta'\beta_i)}{\beta_i(\beta'+\beta-\beta'\beta)} e_j^E + \sum_{j=0}^{i-1} \frac{\beta_j(1-\beta')(\beta_i-\beta)}{\beta_i(\beta'+\beta-\beta'\beta)} e_j^E + \sum_{j'=0'}^{\infty'} \left(1-\frac{\beta}{\beta_i}\right) \frac{\beta'_{j'}}{(\beta'+\beta-\beta'\beta)} e_{j'}^E$$

$$= \sum_{j=i}^{k+m-1} \frac{\beta_j(\beta'+\beta_i-\beta'\beta_i)}{\beta_i(\beta'+\beta-\beta'\beta)} e_j^E + \sum_{j=k}^{i-1} \frac{\beta_j(1-\beta')(\beta_i-\beta)}{\beta_i(\beta'+\beta-\beta'\beta)} e_j^E + \frac{\beta_{k+m}(\beta'+\beta_i-\beta'\beta_i)}{\beta_i(\beta'+\beta-\beta'\beta)} e_{k+m}^E$$

or probabilistically, $h_i^E = P_i$(E is eventually reached)

$$h_i^E = 1 \qquad \text{if } k \le i \le k+m$$

$$= H_{ik} \qquad \text{otherwise.}$$

Similarly if E is a finite set containing the two points 0 and 0', E={n', (n-1)', ..., 0', 0, ..., m} then E is an equilibrium set and

$$e_i^E = \frac{\beta}{\beta_m} \qquad \text{if } i = m$$

$$= \frac{\beta'}{\beta'_{n'}} \qquad \text{if } i = n'$$

$$h_i^E = 1 \qquad \text{if } n' \le i \le m$$

$$= 1 - \frac{\beta}{\beta_i} \qquad \text{if } i > m$$

$$= 1 - \frac{\beta'}{\beta'_{i'}} \qquad \text{if } i' > n'.$$

The capacity of E is in this case a constant independent of n' and m

$$C(E) = \alpha e^E = \beta'_{n'} \frac{\beta'}{\beta'_{n'}} + \beta_m \frac{\beta}{\beta_m} = \beta + \beta'.$$

Infinite sets of the form E={..., (n+1)', n'}$\cup${m, m+1, ...} cannot be equilibrium sets since $h^E = \mathbb{1}$ and hence $e^E = (I-P)h^E = 0$ and h^E is not a potential.

We saw that all subsets of S of the form {n', (n-1)', ..., 0', 0, ..., m-1, m} have capacity $\beta + \beta'$ and hence the supremum is also $\beta + \beta'$ but S is not an equilibrium set. Hence $\hat{N}$ must have columns which don't converge to zero.

Intuitively it is seen that the reverse chain proceeds deterministically a step at a time to the left in S_1(to the right in S_2) until it reaches 0 (0'). From 0 it may step

into any state of S_2 (from 0' it may step in to any state of S_1) with probability $\hat{P}_{0j}$ ($\hat{P}_{0'j}$).

$$\hat{P}_{ij} = \frac{\beta_j}{\beta_i} P_{ji} = \frac{\beta_{i-1}}{\beta_i} p_i = \frac{\beta_i}{\beta_i} = 1 \qquad \text{if } j = i-1 \text{ (i, j both in } S_1)$$

$$= \frac{\beta_j}{\beta_0} q_{j+1} = \beta_j(1-p_{j+1}) = \beta_j - \beta_{j+1} \qquad \text{if } i = 0', j \in S_1$$

$\sum_j \hat{P}_{0'j} = \beta_0 - \beta = 1 - \beta < 1$ and since 0' (or 0) is reached from all states with probability 1, the extended chain for $\hat{P}$ is absorbing and hence the columns of $\hat{N}$ do not tend to zero. This fact is easily seen by looking at

$$\hat{N}_{ij} = \frac{\beta_j}{\beta_i} N_{ji} = \frac{(1-\beta')(\beta_j - \beta)}{\beta' + \beta - \beta'\beta} \qquad \text{if } i < j$$

$$= \frac{\beta' + \beta_j - \beta'\beta_j}{\beta' + \beta - \beta'\beta} \qquad \text{if } i \geq j$$

$$= \frac{\beta_j - \beta}{\beta' + \beta - \beta'\beta} \qquad \text{if } j \in S_1, i' \in S_2.$$

CHAPTER 6

MARTIN BOUNDARIES OF PRODUCTS OF MARKOV CHAINS

6.1. Introduction [26]

Consider two Markov chains (X_n) and (Y_n) with state spaces S_1 and S_2 and transition probabilities $P(X_{n+1} = k / X_n = i) = P_{ik}$ and $P(Y_{n+1} = l / Y_n = j) = P'_{jl}$. Define a Markov chain with state space $S_1 \times S_2$ and transition probabilities $P(ij, kl) = P_{ik}P'_{jl}$. Then $\sum_{kl} P(ij, kl) = \sum_k P_{ik} \sum_l P'_{jl} = 1$, in other words $P(ij,kl)$ form a stochastic matrix. This is called the coupled or the product chain; it describes the joint behavior of a pair of independent systems, each evolving according to the laws of the original Markov chains. It is seen that there exists a Markov chain (X_n, Y_n), $n = 0, 1, \ldots$, having positive initial probabilities and transition probabilities

$$P[\ (X_{n+1}, Y_{n+1}) = (k, l) / (X_n, Y_n) = (i, j)\ ] = P(ij, kl).$$

If both chains are stationary then the product chain is also stationary.

If C_1 and C_2 are two closed sets in S_1 and S_2 (respectively), then obviously $C = C_1 \times C_2$ is also closed in the product chain.

If C is a closed set in the product chain so are $C_1 = \text{proj}_1(C)$ and $C_2 = \text{proj}_2(C)$, where $\text{proj}_i(C)$ is the projection of C on S_i. To see this recall that C is closed if $\sum_{j \in C} P_{ij} = 1 \ \forall i \in C$. Now

$$1 = \sum_{j \in C} P_{ij} \leq \sum_{j1 \in C_1, j2 \in C_2} P_{i1j1} P_{i2j2} = \sum_{j1 \in C_1} P_{i1j1} \sum_{j2 \in C_2} P_{i2j2}$$

$$\leq \sum_{j1 \in C_1} P_{i1j1} \quad \forall i_1 \in C_1.$$

Hence C_1 is closed and by symmetry so is C_2.

The converse is false i.e. if $\text{proj}_1(C)$ and $\text{proj}_2(C)$ are closed C itself need not be closed. We have the following lemma whose proof is trivial.

LEMMA 6. 1. If S_1 and S_2 are reducible so is $S_1 \times S_2$. The converse is false.

Since $P_{(i_1, i_2)}[(X_n, Y_n)=(i_1, i_2) \text{ i.o.}] \leq \min\{P_{i_1}(X_n=i_1 \text{ i.o.}), P_{i_2}(Y_n=i_2 \text{ i.o.})\}$, then if $i = (i_1, i_2)$ is recurrent then i_1 and i_2 are both recurrent in their respective chains. If i_1 or i_2 is transient then $i = (i_1, i_2)$ is transient.

If $j = (j_1, j_2)$, then $E_jT^j \geq E_{j_1}T^{j1}$ and $E_jT^j \geq E_{j_2}T^{j2}$, where E_jT^j is the mean recurrence time of state j, so if j_1 or j_2 is null recurrent so is j in the product chain and if j is positive recurrent so are j_1 and j_2 in their respective chains.

LEMMA 6. 2. Assume that $(P^n)_{ii} > 0$ for some $n \geq 1$. Then a state $i = (i_1, i_2)$ is periodic if and only if at least one of the states i_1 or i_2 is periodic.

PROOF: ($\Rightarrow$) Suppose i is periodic with period $t > 1$, then

$(P_1^n)_{i_1i_1}(P_2^n)_{i_2i_2} = (P^n)_{ii} > 0$ only if $n = kt$, $k = 0, 1, \ldots$.

or, if $n \neq kt$ then $(P_1^n)_{i_1i_1} = 0$ or $(P_2^n)_{i_2i_2} = 0$. We need the following claim to finish the proof.

If a state k is aperiodic, then there exists some N^* such that $(P^n)_{kk} > 0$ for all $n > N^*$.

The proof of the claim is an application of Lemma 5.2 of [4] which says:

Let $E = \{n_1, \ldots, n_r\}$ so that g.c.d. E = 1. Then there exists N^* so that for all $k > N^*$,

$$k = n_1u_1 + \ldots + n_ru_r$$

with the u_i's non-negative integers.

Using the claim, if i_1 and i_2 are both aperiodic, then both $(P_1^n)_{i_1i_1}$ and $(P_2^n)_{i_2i_2}$ are eventually positive. This forces $(P^n)_{ii} = (P_1^n)_{i_1i_1}(P_2^n)_{i_2i_2}$ to be eventually positive, contradicting the periodicity i. Hence at least one of the states is periodic.

($\Leftarrow$) Suppose that one of the states, i_1 for instance, is periodic with period t_1.

Since $(P^n)_{ii} = (P_1^n)_{i_1i_1}(P_2^n)_{i_2i_2} > 0$ only if $n = kt_1$, $k = 0, 1, \ldots$, then i is periodic with period $\geq t_1$.

LEMMA 6. 3. If both chains are ergodic with unique stationary probability distributions π_1 and π_2, then the product chain has a unique stationary probability distribution given by

$$\pi(i) = \pi_1(i_1)\pi_2(i_2) \text{ for } i = (i_1, i_2).$$

PROOF: $$\pi(i) = \pi_1(i_1)\pi_2(i_2) = \sum_k \pi_1(k)P_{ki_1} \sum_u \pi_2(u)P_{ui_2}$$
$$= \sum_{k,u} \pi_1(k)\pi_2(u)P_{ki_1}P_{ui_2}$$
$$= \sum_l \pi(l)P_{li}$$

where $l = (k, u)$ is in the product chain state space. Hence π is stationary, and is unique by the uniqueness of π_1 and π_2.

6.2. Martin boundaries and potentials

A simple example of a product chain can be provided by the product of a renewal chain and a deterministic walk on the set of positive integers. Here the state space is $S = \mathbb{N}\times\mathbb{N}$. Starting at (0, 0) with probability one the process will proceed only to the right and will never leave the region in $\mathbb{R}^2$ between the x-axis and the first bisector y=x. Probability transitions are given by

$$P_{(i_1,i_2)(i_1+1,i_2+1)} = p_{i_2+1},$$

$$P_{(i_1,i_2)(i_1+1,0)} = 1 - p_{i_2+1}, \quad i_1 \geq i_2 \geq 0.$$

Define, as before, $\beta_0 = 1$ and $\beta_i = p_1p_2\ldots p_i$ for $i \geq 1$, and $\beta = \lim_i \beta_i$. Denote L_k, i=0, 1, 2, ..., the line starting at (k, 0) with slope 1. Then for $j \in L_0$,

$$K(i, j) = \frac{H_{ij}}{H_{0j}} = \frac{H_{ij}}{H_{0i}H_{ij}} = \frac{1}{H_{0i}} = \frac{1}{\beta_{i_2}} \qquad \text{if } i = (i_1, i_2) \in L_0$$

$$= 0 \qquad \text{otherwise}$$

Here, and below, 0 refers to the starting state (0, 0).

For $j \in L_k$, $k \neq 0$

$$K(i,j) = \frac{H_{ij}}{H_{0j}} = \frac{H_{ij}}{H_{0i}H_{ij}} = \frac{1}{H_{0i}} = \frac{1}{H_{0(k,0)}\beta_i} \qquad \text{if } i = (i_1, i_2) \in L_k$$

$$= \frac{H_{i(k,0)}H_{(k,0)j}}{H_{0(k,0)}H_{(k,0)j}} = \frac{H_{i(k,0)}}{H_{0(k,0)}} \qquad \text{if } i \in L_m,\ m<k$$

$$= 0 \qquad \text{otherwise.}$$

Thus, $K(., j_n)$ is Cauchy if and only if the sequence of states $\{j_n\}$ is contained in some L_k, and we denote the limit by ∞_k. Since $K(i, \infty_k) > 0$ if and only if $i \in \bigcup_{0\le j\le k} L_j$, it is clear that $\{K(., \infty_k)\}_{k=0}^{\infty}$ are distinct functions and hence $\{\infty_k\}_{k=0}^{\infty}$ are distinct boundary points. Thus the Martin exit boundary is in a one-to-one correspondence with $\mathbb{N}$.

Using the representation of Π, it is seen that harmonic measure is

$$\mu(\infty_0) = \beta,$$

$$\mu(\infty_k) = \beta H_{(0,0)(k,0)}, \quad k \neq 0.$$

6.2.1. Product of a renewal chain by itself

THEOREM 6.1. The Markov chain resulting from the product of renewal chain by itself possesses a countably infinite number of boundary exit points.

PROOF: Let us consider again the transient renewal Markov chain on the non-negative integers with transition probabilities

$$P_{i-1,i} = p_i, \ i > 0, \ P_{i-1,0} = q_i = 1 - p_i, \ P_{0,1} = 1$$

and the values defined by

$$\gamma_0 = 1, \ \gamma_i = p_i\gamma_{i-1} = \Pi_{k=1}^{i} p_k, \text{ and } \gamma = \lim_{i \to \infty} \gamma_i > 0.$$

The chain resulting from the product is defined on the set of lattice points $S = \{(i, j) \in \mathbb{N}\times\mathbb{N}\}$, and has transition probabilities

$$P_{(i_1-1,i_2-1)(i_1,i_2)} = p_{i_1}p_{i_2}$$

$$P_{(i_1-1,i_2-1)(0,0)} = q_{i_1}q_{i_2} = (1 - p_{i_1})(1 - p_{i_2})$$

$$P_{(i_1-1,i_2-1)(i_1,0)} = p_{i_1}q_{i_2} = p_{i_1}(1 - p_{i_2})$$

$$P_{(i_1-1,i_2-1)(0,i_2)} = q_{i_1}p_{i_2} = (1-p_{i_1})p_{i_2}.$$

Denote by 0 or 0_0 the state (0, 0), by 0_k any state (k, 0) for $k \geq 1$ and by 0_{-k} the state (0, k) for $k \geq 1$. Let L_k denote the ray starting from 0_k with slope 1 for $k \in \mathbb{Z}$. Let $\beta_{j,k}$ be the probability that beginning at 0_k, the chain remains in L_k for at least j steps. Then for $j \in L_0$

$$K(i, j) = \frac{H_{ij}}{H_{0j}} = \frac{H_{ij}}{H_{0i}H_{ij}} = \frac{1}{H_{0i}} \qquad \text{if } 0 < i < j$$

$$= \frac{H_{i0}H_{0j}}{H_{0j}} = H_{i0} \qquad \text{if } i \notin L_0, \text{ or } i \in L_0 \text{ and } i \geq j.$$

Since K(i, j) is independent of j for large j, the set of states supported by L_0 is a Cauchy sequence to which we attach the limit ∞_0. Moreover $K(i, \infty_0)$ is normalized (i.e. $\pi K=1$) and regular.

Similarly if $j \in L_k$,

$$K(i, j) = \frac{H_{ij}}{H_{0j}} = \frac{H_{ij}}{H_{0i}H_{ij}} = \frac{1}{H_{0i}} \qquad \text{if } i \in L_k$$

$$= \frac{H_{i,0_k}H_{0_k,j}}{H_{0,0_k}H_{0_k,j}} = \frac{H_{i,0_k}}{H_{0,0_k}} \qquad \text{otherwise.}$$

Hence, by the same argument as for L_0, the set of all the states belonging to L_k form a Cauchy sequence.

Since

$$\lim_{i \to \infty, i \in L_0} K(i, \infty_0) = \frac{1}{H_{0\infty_0}}$$

$$\lim_{i \to \infty, i \in L_0} K(i, \infty_k) = \lim_{i \to \infty} \frac{H_{i,0_k}}{H_{0,0_k}} = 0$$

then

$$K(i, \infty_0) \neq K(i, \infty_k) \text{ for all } k \neq 0.$$

Also,

$$\lim_{i \to \infty, i \in L_k} K(i, \infty_k) = \lim_{i \to \infty, i \in L_k} \frac{1}{H_{0i}} = \frac{1}{H_{0\infty_k}}$$

$$\lim_{i \to \infty, i \in L_k} K(i, \infty_{k'}) = \lim_{i \to \infty, i \in L_k} \frac{H_{i,0_{k'}}}{H_{0,0_{k'}}} = 0 \qquad \text{for all } k \neq k'.$$

Hence to each ray to infinity starting at some 0_k corresponds a distinct regular function $K(., \infty_k)$ which means that, since $k \in \mathbb{Z}$, the set of Cauchy sequences in S^* is in a one-to-one correspondence with $\mathbb{Z}$.

Harmonic measure is given by the representation of the function Π:

$$\begin{pmatrix} \frac{1}{H_{0,i0}} & \frac{H_{i0,0_1}}{H_{0,0_1}} & \frac{H_{i0,0_2}}{H_{0,0_2}} & \frac{H_{i0,0_3}}{H_{0,0_3}} \cdots \\ H_{i_1,0} & \frac{1}{H_{0,i_1}} & \frac{H_{i_1,0_2}}{H_{0,0_2}} & \frac{H_{i_1,0_3}}{H_{0,0_3}} \cdots \\ H_{i2,0} & \frac{H_{i2,0_1}}{H_{0,0_1}} & \frac{1}{H_{0,i_2}} & \frac{H_{i_2,0_3}}{H_{0,0_3}} \cdots \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{pmatrix} \begin{pmatrix} \mu(\infty_0) \\ \mu(\infty_1) \\ \mu(\infty_2) \\ \mu(\infty_3) \\ . \\ . \\ . \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ . \\ . \\ . \end{pmatrix} \qquad (6.1)$$

Using Proposition 5.3 we find that k^{th} row of the left hand matrix is dominated by $\frac{1}{H_{0,\infty_k}}$, where $H_{0,\infty_k} := \mathrm{Lim}_{j \in L_k,\, j \to \infty} H_{0j}$. Using the result of section 2.1 we can conclude that

$$\begin{pmatrix} \frac{1}{H_{0\infty_0}} & 0 & 0 & 0 \cdots \\ 0 & \frac{1}{H_{0\infty_1}} & 0 & 0 \cdots \\ 0 & 0 & \frac{1}{H_{0\infty_2}} & 0 \cdots \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{pmatrix} \begin{pmatrix} \mu(\infty_0) \\ \mu(\infty_1) \\ \mu(\infty_2) \\ \mu(\infty_3) \\ . \\ . \\ . \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ . \\ . \\ . \end{pmatrix}$$

So for each $k \in \mathbb{Z}$, $\mu(\infty_k) = H_{0,\infty_k}$.

Harmonic measure is putting less and less mass on ∞_k as k gets far from 0 since the starting measure π puts all its mass on the state 0.

Any finite set or infinite strip $E=\{(i, j): 0 \le i \le \infty, 0 \le j \le m$, or $0 \le j \le \infty$,

$0 \le i \le m, m \in \mathbb{N}\}$ is an equilibrium set with equilibrium potential

$$h_i^E = 1 \qquad \text{if } i \in E$$

$$= \sum_{j \in \partial E} H_{ij} \qquad \text{if } i \in S\text{-}E,$$

where $\partial E = \{j \in E: \exists k \in S\text{-}E, P_{kj} > 0\}$.

The charge is given by

$$e_i^E = 1 - \sum_{j \in E} H_{ij} \qquad \text{if } i \in \partial^* E$$

$$= 0 \qquad \text{Otherwise,}$$

where $\partial^* E = \{i \in E: \exists k \in S\text{-}E, P_{ik} > 0\}$.

If β is the reference measure the capacity is given by

$$C(E) = \sum_{i \in S} \beta_i e_i^E = \sum_{i \in \partial^* E} \beta_i (1 - \sum_{j \in E} H_{ij}).$$

S is not an equilibrium set because the extended chain for $\hat{P}$ is absorbing

$$\hat{P}_{jj-1} = \frac{\beta_{j-1}}{\beta_j} P_{j-1,j} = \frac{\beta_j}{\beta_j} = 1$$

$$\hat{P}_{0(j_1,j_2)} = \frac{\gamma_{j_1}\gamma_{j_2}}{\gamma_0\gamma_0} P_{(j_1,j_2)0} = \gamma_{j_1}\gamma_{j_2}(1-p_{j_1+1})(1-p_{j_2+1})$$

$$= (\gamma_{j_1} - \gamma_{j_1+1})(\gamma_{j_2} - \gamma_{j_2+1}) \qquad \text{for all } j.$$

$$\hat{P}_{0_k(j_1,j_2)} = \gamma_{j_1}\gamma_{j_2} P_{(j_1,j_2)0_k} = \gamma_{j_1}\gamma_{k-1}(1-p_{j_1+1})p_k$$

$$= \gamma_k(\gamma_{j_1} - \gamma_{j+1}) \qquad \text{if } j = (j_1, k-1) \text{ and } k \ge 1$$

$$= 0 \qquad \text{otherwise.}$$

Thus the reverse chain proceeds deterministically a step at a time along a line L_k until it reaches 0_k. From 0_k it may step into any state $(j_1, k-1)$ and does so with probability

$$\hat{P}_{0_k(j_1,k-1)} = \gamma_k(\gamma_{j_1}-\gamma_{j+1}).$$

Along the line L_0, once 0 is reached, the process may step into any state j and does so with probability

$$\hat{P}_{0(j_1,j_1)} = (\gamma_{j_1}-\gamma_{j_1+1})(\gamma_{j_2}-\gamma_{j_2+1}).$$

Since

$$\sum_j \hat{P}_{0j} = \sum (\gamma_{j_1}-\gamma_{j_1+1})(\gamma_{j_2}-\gamma_{j_2+1}) = (1-\gamma)^2 < 1,$$

$$\sum_j \hat{P}_{0_k j} = \sum_{j_1} \hat{P}_{0_k(j_1,k-1)} = \sum_{j_1} \gamma_k(\gamma_{j_1}-\gamma_{j+1}) = \gamma_k(1-\gamma) < 1,$$

and since from any state $j \in L_k$, 0_k is reached with probability one, the extended chain for $\hat{P}$ is absorbing.

6.2.2. Product of renewal chain and a symmetric random walk with reflecting barriers

Let the state space of the random walk contain only two points {0, 0'} with transition probabilities $P_{00'} = P_{0'0} = 1$. The state space of the product chain is $S = \mathbb{N} \times \{0, 0'\} = \{(0, 0), (1, 0), (2, 0), ..., (0, 0'), (1, 0'), (2, 0'), ...\} = \{0, 1, 2, ..., 0', 1', 2', ...\}$. Let $S_1 = \{0, 1', 2, 3', ...\}$, $S_2 = \{0', 1, 2', 3, ...\}$ and the starting distribution be concentrated at (0, 0). Then the process starts in S_1 and either keeps on moving to the right without leaving S_1, or jumps back to (0, 0) and restarts again in S_1, or jumps back to 0' in S_2 and behave exactly as in S_1. The process is infinitely often in either S_1 or S_2. The transition probabilities are

$$P_{j,(j+1)'} = P_{j',j+1} = p_{j+1}$$

$$P_{j,0'} = P_{j',0} = q_{j+1} = 1 - p_{j+1}, \quad j = 0, 1, ...$$

THEOREM 6. 2. The Martin exit boundary of the Markov chain obtained from the product of a renewal chain and a two-state symmetric random walk with reflecting barriers contains two points.

PROOF: Let $\omega_{i0} = P_i((0,0)$ is visited before $(0,0'))$, and $\omega_{i0'} = P_i((0,0')$ is visited before $(0,0))$

For $j \in S_1$ we have $H_{0j} = \beta_j + \omega_{00}H_{0j} + \omega_{00'}H_{0'j} = \beta_j + \omega_{00}H_{0j} + \omega_{00'}H_{0'0}H_{0j}$.

Hence

$$H_{0j} = \frac{\beta_j}{1 - \omega_{00} - \omega_{00'}H_{0'0}}$$

$$H_{ij} = \frac{\beta_j}{\beta_i} + \omega_{i0}H_{0j} + \omega_{i0'}H_{0'j} = \frac{\beta_j}{\beta_i} + \omega_{i0}H_{0i}H_{ij} + \omega_{i0'}H_{0'i}H_{ij} \qquad \text{if } i \in S_1$$

$$= H_{i0}H_{0j} \qquad \text{if } i \in S_2$$

or

$$H_{ij} = \frac{\beta_j}{\beta_i(1-\omega_{i0}H_{0i} - \omega_{i0'}H_{0'i})} \qquad \text{if } i \in S_1$$

$$= H_{i0}H_{0j} \qquad \text{if } i \in S_2$$

Therefore,

$$K(i, j) = \frac{H_{ij}}{H_{0j}} = \frac{1-\omega_{00}-\omega_{00'}H_{0'0}}{\beta_i(1-\omega_{i0}H_{0i} - \omega_{i0'}H_{0'i})} \qquad \text{if } i \in S_1$$

$$= H_{i0} \qquad \text{if } i \in S_2.$$

For $j \in S_1$,

$$H_{0j} = H_{00'}H_{0'j} = H_{00'}\frac{\beta_j}{(1-\omega_{0'0'} - \omega_{0'0}H_{00'})}$$

$$H_{ij} = H_{i0'}H_{0'j} \qquad \text{if } i \in S_1$$

$$= \frac{\beta_j}{\beta_i(1-\omega_{i0'}H_{0'i} - \omega_{i'0}H_{0i})} \qquad \text{if } i \in S_2.$$

Therefore,

$$K(i, j) = \frac{H_{ij}}{H_{0j}} = \frac{H_{i0'}}{H_{00'}} \qquad \text{if } i \in S_1$$

$$= \frac{1-\omega_{0'0'}-\omega_{0'0}H_{00'}}{\beta_i H_{00'}(1-\omega_{i0'}H_{0'i} - \omega_{i'0}H_{0i})} \qquad \text{if } i \in S_2$$

It is clear that S_1 and S_2 are both Cauchy sequences with limits ∞_1 and ∞_2 (respectively).

Now,

$$\lim_{i \to \infty,\, i \in S1} K(i, \infty_1) = \frac{1-\omega_{00}-\omega_{00'}H_{0'0}}{\beta} \neq \lim_{i \to \infty,\, i \in S1} K(i, \infty_2) = 0$$

Hence the boundary consists of the two points $\{\infty_1, \infty_2\}$.

It is easy to see that harmonic measure μ assigns the following weight to ∞_1 and ∞_2

$$\mu(\infty_1) = \frac{H_{00'}\beta}{1-\omega_{0'0'}-\omega_{0'0}H_{00'}}$$

$$\mu(\infty_2) = \frac{\beta}{1-\omega_{00}-\omega_{00'}H_{0'0}}.$$

Since $1-\omega_{0'0'}-\omega_{0'0}H_{00'} = 1-\omega_{00}-\omega_{00'}H_{0'0} = \beta+H_{00'}\beta$, then

$$\mu(\infty_1)+\mu(\infty_2) = 1.$$

THEOREM 6. 3. The Martin exit boundary of the Markov chain obtained from the product of a renewal chain and an ergodic random walk with reflecting barriers contains two points.

To prove this theorem we are going to use the notion of *sojourn* sets as given by Feller[11]. Roughly, a sojourn set in the state space of a Markov chain is a subset which, after a finite number of steps, the chain may enter and never exit from. It is shown that to each sojourn set corresponds a Feller boundary point; and when the number of boundary points is finite it coincides with the Martin boundary. Hence our task is to identify all sojourn sets which give rise to distinct boundary points. We start looking at the very simple example of the product of an aperiodic, communicating two-state random walk and a deterministic walk on the non-negative integers. Thus $Z_n = (X_n, Y_n)$ where X_n is an ergodic Markov chain on two states, 0 and 1, and Y_n is a deterministic time, i.e., $Y_n = n$. It is intuitively clear that the product chain has one boundary point, so that we will attempt to prove that there is only one sojourn set, namely the whole state space. Now,

Let $0 = t_0 < t_1 < \ldots$ be an arbitrary sequence of times. We have to show that $P_1(X_{t_k} = 1\ \forall k) = 0$.

$$P_1(X_{t_k} = 1\ \forall k) = P_1(\bigcap_{k\geq 1}\{X_{t_k} = 1\})$$

$$= \prod_{i\geq 0} P_{11}(t_{i+1} - t_i) \qquad \text{by the Markov property}$$

$$= \prod_{i\geq 0}(1 - P_{10}(t_{i+1} - t_i)).$$

Since the random walk is aperiodic, and $0 \leftrightarrow 1$, $P_{10}(n)$ converges to a non-zero constant as $n\to\infty$. Therefore the above product is zero. Since the sequence of times is arbitrary, this shows that there is no infinite subset of the state space which is a sojourn set except the whole state space which proves the result.

If the random walk X_n has an arbitrary finite state space S, then we must show that for any infinite subset A of the state space $E = S \times \mathbb{N}$, E-A cannot be a sojourn set or $P_i[\exists n: (X_n, n)\in A] = 1$ it suffices to show that for any $s^*\in S$, and any sequence of times $0 = t_0 < t_1 < \ldots$, we have $P_i[\exists k: X_{t_k} = s^*] = 1$, or $P_i[\forall k: X_{t_k} \in S - \{s^*\}] = 0$. We may assume, without loss of generality that $t_{k+1} - t_k$ exceeds the number of states in S for each k.

Let $S - \{s^*\} := S'$, then

$$P_i[\forall k: X_{t_k} \in S'] = P_i(\bigcap_{k\geq 1}\{X_{t_k} \in S'\}) =$$

$$= P_i(X_{t_1}\in S') \prod_{k\geq 1} P_i(X_{t_{k+1}}\in S' / X_{t_k} \in S') \quad \text{by the Markov property}$$

$$= P_i(X_{t_1}\in S') \prod_{k\geq 1} P_i(X_{t_{k+1}}\in S' / \bigcup_{s\in S'}\{X_{t_k}= s\})$$

$$= P_i(X_{t_1}\in S') \prod_{k\geq 1} \frac{\sum_{s\in S'} P_i(X_{t_{k+1}}\in S' / X_{t_k}= s)P_i(X_{t_k}= s)}{\sum_{s\in S'} P_i(X_{t_k}= s)}$$

$$= P_i(X_{t_1}\in S') \prod_{k\geq 1} \frac{\sum_{s\in S'} P_s(X_{t_{k+1}-t_k}\in S')P_i(X_{t_k}= s)}{\sum_{s\in S'} P_i(X_{t_k}= s)},$$

using the Markov property.

To finish the proof we show that the quantity on the right hand side of the product sign is bounded away from 1 so that the product is zero.

Since X_n is a random walk, if k is greater than the number of states in S, then $\sup_j P_j(X_k\in S') < 1$. Therefore for $k \leq n$,

$$P_i(X_n \in S') = \sum_{j\in S} P_i(X_n\in S' / X_{n-k}= j)P_i(X_{n-k}= j)$$

$$= \sum_{j\in S} P_j(X_k \in S') P_i(X_{n-k} = j)$$

$$\leq \rho \sum_{j\in S} P_i(X_{n-k} = j) \leq \rho < 1.$$

Hence $\sup_{n\geq k} \sup_{i\in S} P_i(X_n \in S') < 1$ which implies that

$$P_i[\forall k: X_{t_k} \in S'] \leq P_i(X_{t_1} \in S') \prod_{k\geq 1} \frac{\sum_{s\in S'} P_i(X_{t_k} = s)}{\sum_{s\in S'} P_i(X_{t_k} = s)} = 0.$$

Consequently the whole state space is the only sojourn set and we can say that the product of an ergodic finite state space random walk and a deterministic walk on the non-negative integers possesses one Feller boundary point and hence one Martin boundary point.

PROOF OF THOREM 6. 3. Denote by E the product state space, by S_1 the state space of the renewal chain and by S_2 that of the random walk. Let A be an arbitrary but infinite subset of E, $\tau_A = \inf\{n \geq 1: (X_n, Y_n) \in A\}$ which represents the first time the subset A is entered and $\tau = \inf\{n \geq 1: X_n = 0\}$ which is the first time the renewal chain returns back to zero. We want to show that $P_i(\tau_A < \infty) = 1$, $i \in S_2$.

Let $P_{i*}(\tau_A < \infty) = \inf_{i\in S_2} P_i(\tau_A < \infty)$ which exists as S_2 is finite. Now,

$$P_{i*}(\tau_A < \infty) = P_{i*}(\tau_A < \infty, \tau = \infty) + P_{i*}(\tau_A < \infty, \tau < \infty)$$

$$= P_{i*}(\tau_A < \infty, \tau = \infty) + P_{i*}(\tau_A \leq \tau < \infty) + P_{i*}(\tau < \tau_A < \infty). \qquad (6.2)$$

First of all, we see that $P_{i*}(\tau_A < \infty, \tau = \infty) = P_{i*}(\exists n: (X_n, Y_n) \in A;\ X_k \neq 0\ \forall k)$

$$= P_{i*}(\exists n: (n, Y_n) \in A;\ X_k \neq 0\ \forall k),$$

because on the set $\{X_k \neq 0\ \forall k\}$ we have $X_n = n$ for all n.

Since X_n and Y_n are independent, the probability above is equal to,

$$P_{i*}(\exists n: (n, Y_n) \in A) P_{i*}(X_k \neq 0\ \forall k),$$

and using the previous example this is,

$$P_{i*}(X_k \neq 0\ \forall k) = P_{i*}(\tau = \infty).$$

Considering the third term of (6.1) we have

$$P_i^*(\tau < \tau_A < \infty) = \int E(1_{\{\tau_A \circ \theta\tau < \infty,\, \tau < \infty,\, \tau < \tau_A\}} / \mathcal{B}_\tau)\, dP_i^*$$

$$= \int_{\{\tau < \infty, \tau < \tau_A\}} P_{X_\tau}(\tau_A < \infty)\, dP_i^*$$

$$\geq P_i^*(\tau_A < \infty) P_i^*(\tau < \infty,\, \tau < \tau_A),$$

by the definition of i*.

Putting these together gives,

$P_i^*(\tau_A < \infty) \geq P_i^*(\tau = \infty) + P_i^*(\tau_A \leq \tau < \infty) + P_i^*(\tau_A < \infty) P_i^*(\tau < \infty,\, \tau < \tau_A)$,

or $P_i^*(\tau_A < \infty) \geq 1$ after solving the inequality and noting that

$P_i^*(\tau = \infty) + P_i^*(\tau_A \leq \tau < \infty) + P_i^*(\tau < \infty,\, \tau < \tau_A) = 1$. Therefore, $P_i(\tau_A < \infty) = 1$

and we can conclude that there is only one sojourn set, E itself, to which corresponds one Feller boundary point and hence one Martin boundary point.

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